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A Multicriteria Decision-making Approach to Create Accessible Environments to Empower Mobility-impaired Individuals

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ABSTRACT

Individuals with mobility disabilities can experience numerous health advantages when connecting with nature in various ways, such as passive enjoyment, active participation, or rehabilitative programs. These benefits encompass physical and mental benefits, as well as social gains. However, a range of concerns related to making natural environments accessible to and usable by people with mobility impairments demand the attention of various professionals, including caregivers, landscape architects, rehabilitation therapists, and policymakers. Efforts to promote inclusivity and accessibility aim to remove barriers and create environments where individuals with disabilities can participate fully in education, employment, public life, and social activities. This may involve adapting physical spaces, promoting awareness, providing assistive technology, offering support services, and understanding of disability issues. To enhance the accessibility of public places for disabled people, we must consider multiple criteria and risks. In this article, to address such issues we develop three multicriteria decision-making (MCDM) approaches based on picture cubic fuzzy information. Since aggregation operators (AOPs) play a crucial role in decision-making, we present the aggregation proficiency for picture cubic fuzzy information and develop a series of AOPs, such as picture cubic fuzzy Hamacher order weighted averaging (PCFHOWA), picture cubic fuzzy Hamacher weighted averaging (PCFHWA), picture cubic fuzzy Hamacher hybrid averaging (PCFHHA) operators and present some essential properties of these Opts. After studying their fundamental operations and properties, we utilize these operators to develop multicriteria decision making (MCDM) model with picture cubic fuzzy information. We present the extended TOPSIS method and extended VIKOR mothod for MCDM problems. We present a numerical example related to improving accessibility for disabled people in a public park. The results explore the effectiveness of our proposed methodologies and provide accurate measures to address the uncertainty related to the accessibility of disabled people to public places.

KEYWORDS

accessibility, barriers, disabilities, TOPSIS method, picture cubic fuzzy set, VIKOR method

INTRODUCTION

Mobility impairment is a broad category of disability that includes individuals with diverse physical disabilities. This disability category includes conditions such as upper or lower limb impairments, reduced manual dexterity, and difficulties coordinating various body organs (Pedzisai and Charamba, 2023). Mobility disabilities can be present from birth (congenital) or develop later in life (acquired), sometimes due to diseases or injuries. People with broken skeletal structures also fall into this category. Individuals with physical impairments often rely on assistive devices like crutches, canes, wheelchairs, and prosthetic limbs to enhance their mobility. According to the World Health Organization's report on March 7, 2023 (https://www.who.int/news-room/fact-sheets/detail/disability-and-health), approximately 1.3 billion individuals grapple with significant disabilities, constituting about 16% of the global population, which translates to roughly one in every six people (Kuper et al., 2024). Some individuals with disabilities face a life expectancy that is up to 20 years shorter compared to those without disabilities (DuBois et al., 2024). Furthermore, they carry twice the risk of developing various health conditions such as depression, asthma, diabetes, stroke, obesity, or oral health issues (Dorsey Holliman et al., 2023). Persons with disabilities encounter significant difficulties, approximately 15 times more, in accessing transportation due to issues of affordability and accessibility when compared to those without disabilities. It is crucial to consider the needs of individuals with disabilities when planning for and responding to health emergencies because they are more susceptible to both direct and indirect impacts (Waitt et al., 2024). For instance, during the COVID-19 pandemic, those with disabilities living in institutions experienced social isolation and reports of residents being overmedicated, sedated, or confined, with instances of self-harm also being reported (Brennan, 2020).

Improving accessibility for individuals with mobility impairments typically involves making physical spaces, facilities, and services more accommodating and barrier-free, so these individuals can navigate and participate in various activities with greater ease and independence (Apostolidou and Fokaides, 2023). This might include features like ramps, elevators, accessible parking, wider doorways, and more to ensure that everyone can access and enjoy public spaces and services (Pettersson et al., 2023). It is important to note that the experience of disability varies widely among individuals. Some disabilities may be well managed with assistive devices or treatments, allowing individuals to lead fulfilling lives, while others may face significant challenges in their daily lives (Benham et al., 2023). Efforts to promote inclusivity and accessibility aim to remove barriers and create environments where individuals with disabilities can participate fully in education, employment, public life, and social activities. This may involve adapting physical spaces, providing assistive technology, offering support services, and promoting awareness and understanding of disability issues (Grimmett et al., 2023). In this article, we will present a multicriteria decision-making (MCDM) example related to improving accessibility for the disabled people in a public park.

Using multicriteria group decision-making (MCGDM) approaches, it is possible to rank the items in a problem and select the best option. Decision sciences heavily depend on MCGDM. Evaluation data for various criteria, provided by decision makers (DMs), are used to choose the most suitable alternative (Altr) from a set of finite options (Jana et al., 2023; Khan et al., 2023). In many decision-making problems and hesitant situations, experts often find it challenging to express their opinions with crisp values and struggle to determine exact values for potential Altrs when dealing with conflicting criteria or attributes. In 1965, Zadeh (1965) introduced the concept of fuzzy sets (FSs) as a solution for addressing problems in uncertain conditions. FSs provide a basis for handling uncertain assessments, but they have limitations in representing non-membership. Atanassov (1999) extended the concept of FS into intuitionistic fuzzy sets (IFSs), offering a more comprehensive framework. However, IFS alone cannot completely address the challenges of uncertainty. To tackle these issues, Jun et al. (2011) introduced the concept of cubic set (CS) specifically designed to deal with problems of uncertainty. CSs provide a more elaborate way of representing and managing uncertainty in decision-making problems (Muneeza et al., 2020, 2022). Unlike traditional FSs, the CS theory clarifies the differentiation between unpredictable, unsatisfied, and satisfied information (Qiyas et al., 2021). This differentiation can be especially useful in cases where standard FSs are insufficient in capturing the complexity of the data (Muneeza and Abdullah, 2020; Muneeza et al., 2023). When compared to FS, CS has more alluring data (Kaur and Garg, 2018a,b). In traditional IFSs and CSs, only two types of responses are considered: "yes" and "no." However, when dealing with selection problems, there are instances where three types of responses are required, namely "yes," "no," and "refusal." Handling the "refusal" response can be particularly challenging. To address this limitation, Cuong (2013) introduced a novel concept known as picture fuzzy sets (PFSs). PFSs provide a more comprehensive framework by distinguishing between positive, neutral, and negative membership grades using three distinct functions. By using the CS theory, Khoshaim et al. (2021), have introduced a new approach of PFS through application of the CS theory and built up the notion of picture cubic fuzzy set (PCFS), in which every element comprises the positive, negative, and neutral membership functions. PCFS is a hybrid set which can have substantially too much data to communicate a PFS and CS simultaneously for dealing the vulnerabilities in the information. Since aggregation operators (AOPs) play a crucial role in decision-making, we present the aggregation proficiency for PCF information and develop a series of AOPs, such as PCFHOWA operator (Opts), PCFHWA Opt, and PCFHHA Opt and present some fundamental characteristics of the developed Opts. When applied to genuine ㄱ, Hamacher Opts display more exact outcomes depending upon the PCF data.

Hwang and Yoon first proposed the "technique for order preference by similarity to ideal solution" (TOPSIS) approach in 1981 (Hwang and Yoon, 1981). This approach was later extended by many authors. The TOPSIS method is specially used in complicated decision-making problems. For the selection of Altrs, the TOPSIS method is a very effective tool (Jahanshahloo et al., 2006). The VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method was presented by Opricovic (1998) for the solution of MCDM problems. The conventional VIKOR approach was expanded upon by Liao and Xu (2013) to include uncertain fuzzy environments. Park et al. (2011) presented the VIKOR method with interval-valued intuitionistic fuzzy numbers and applied it in decision-making problems. Supply chain management (Nazam et al., 2020), design (Ighravwe and Oke, 2020), medical diagnostics (Erdebilli et al., 2023), risk management (Sun et al., 2020), logistics (Wang et al., 2021), engineering (Li et al., 2020), building, and transportation (Khan et al., 2024) are just a few of the many industries where the VIKOR approach is used. TOPSIS is based on the idea that the best choice should be as far away as possible from the negative ideal solution (NIS) while being as close as possible to the positive ideal solution (PIS) (Francy and Rao, 2024). This approach is favored by risk-averse DMs who want to make choices that are both low in risk and highly advantageous. The VIKOR method calculates the ideal point based on a specific measure of "closeness" to the PIS. This method is suitable for situations where the DM aims to minimize risk to an extreme extent while seeking high benefits. In this paper, we have extended the concept of the TOPSIS and VIKOR methods for PCF numbers (PCFNs). As a prevalent set, PCFNs show uncommon execution in giving reliable, unclear, and vague assessment information due to changed and loosened up conditions. Subsequently, PCFNs may be a good methodology for surveying the capability of options.

Structure of our proposed work

We have provided a structured outline of the different sections in our article. Here is a simplified version: the first section contains the introduction. A few fundamental concepts are presented in the Preliminaries section. In the Picture Cubic Fuzzy Hamacher Averaging AOPs section, we discuss basic Hamacher operations based on PCFNs and explore averaging AOPs like picture cubic fuzzy Hamacher weighted averaging (PCFWA) Opt, PCFOW Opt, and PCFHHA Opt, along with some of their important properties. In the MCDM Algorithm for Picture Cubic Fuzzy AOPs section, we construct a step-by-step algorithm for handling MCGDM issues in the context of PCFNs. In the Algorithms for Decision-making section, we define two MCDM algorithms, i.e. the TOPSIS method and the VIKOR method. The Numerical Application section employs the MCGDM approach to an example related to improving accessibility for disabled people in a public park using PCF Hamacher (AOPs). In the Comparison Analysis section, we compare the decision findings of our proposed strategies to the existing techniques to demonstrate the effectiveness and practicality of our MCGDM approach to reflect the applicability and efficiency of the recognized MCGDM approach. Finally, the Conclusion section summarizes our work and findings.

PRELIMINARIES

In this chapter, we briefly review the basic concepts associated with PCFS and their significant properties.

Definition 1. (Khoshaim et al., 2021) *Let ∃ be a non-empty set, then PCFS P in ∃, is given as follows:*

$$P = \{v, \langle c_I, c_I', c_I'' \rangle \mid v \in \exists\},$$

or

$$P = \{(v, \langle [\dot{z}^-, \dot{z}^+], \vartheta \rangle, \langle [\check{s}^-, \check{s}^+], \delta \rangle, \langle [\dot{c}^-, \dot{c}^+], \dot{e} \rangle) \mid v \in \exists\},$$

where $\langle [\dot{z}^-, \dot{z}^+], \vartheta \rangle$ *denotes the membership grade* $\langle [\check{s}^-, \check{s}^+], \delta \rangle$ *and* $\langle [\dot{c}^-, \dot{c}^+], \dot{e} \rangle$ *denotes the non-membership grade of P. Here* $[\dot{z}^-, \dot{z}^+] \subset [0, 1]$, $[\check{s}^-, \check{s}^+] \subset [0, 1]$, $[\dot{c}^-, \dot{c}^+] \subset [0, 1]$, $\dot{e} : \exists \to [0, 1]$, $\delta : \exists \to [0, 1]$ *and* $\vartheta : \exists \to [0, 1]$ *subject to* $\vartheta + \delta + \dot{e} \leq 1$ *and* $\sup[\dot{z}^-, \dot{z}^+] + \sup[\check{s}^-, \check{s}^+] + \sup[\dot{c}^-, \dot{c}^+] \leq 1$. *Also,*

$$\pi(P) = \{\langle [1, 1] - [[\dot{z}^-, \dot{z}^+] + [\check{s}^-, \check{s}^+] + [\dot{c}^-, \dot{c}^+]] \rangle, \langle 1 - (\vartheta + \delta + \dot{e}) \rangle\},$$

$$\pi(P) = \{[1 - (\dot{z}^- + \check{s}^- + \dot{c}^-), 1 - (\dot{z}^+ + \check{s}^+ + \dot{c}^+), 1 - (\vartheta + \delta + \dot{e})]\},$$

called PCFS hesitation margin of $v \in \exists$ *for PCFS. The pair* $([\dot{z}^-, \dot{z}^+], \vartheta, [\check{s}^-, \check{s}^+], \delta, [\dot{c}^-, \dot{c}^+], \dot{e})$ *is called the PCF value (PCFV) or PCFN and is denoted by P, i.e.* $P = ([\dot{z}^-, \dot{z}^+], \vartheta, [\check{s}^-, \check{s}^+], \delta, [\dot{c}^-, \dot{c}^+], \dot{e})$.

Definition 2. (Khoshaim et al., 2021) *Let* $P = (\langle [\dot{z}^-, \dot{z}^+], \vartheta \rangle, \langle [\check{s}^-, \check{s}^+], \delta \rangle, \langle [\dot{c}^-, \dot{c}^+], \dot{e} \rangle)$ *be a PCFN. Then, score function of P is defined as follows:*

$$S(P) = [(\dot{z}^- + \dot{z}^+ + \vartheta + \check{s}^- + \check{s}^+ + \delta - \dot{c}^- - \dot{c}^+ - \dot{e}) / 3],$$

such that $S(P) \in [-1, 1]$.

Definition 3. (Khoshaim et al., 2021) *Let P be a PCFN, the accuracy function of P is given as follows:*

$$H(P) = [(\dot{z}^- + \dot{z}^+ + \vartheta + \check{s}^- + \check{s}^+ + \delta + \dot{c}^- + \dot{c}^+ + \dot{e}) / 3],$$

where $H(P) \in [0, 1]$.

Definition 4. (Khoshaim et al., 2021) *Let* $P_1 = \{(v, \langle [\dot{z}_1^-, \dot{z}_1^+], \vartheta_1 \rangle, \langle [\check{s}_1^-, \check{s}_1^+], \dot{e}_1 \rangle, \langle [\dot{c}_1^-, \dot{c}_1^+], \dot{e}_1 \rangle) \mid v \in \exists\}$ *and* $P_2 = \{(v, \langle [\dot{z}_2^-, \dot{z}_2^+], \vartheta_2 \rangle, \langle [\check{s}_2^-, \check{s}_2^+], \dot{e}_2 \rangle, \langle [\dot{c}_2^-, \dot{c}_2^+], \dot{e}_2 \rangle) \mid v \in \exists\}$, *be two PCFNs; their scores are* $S(P_2)$ *and* $S(P_1)$ *and the accuracy functions are* $H(P_2)$ *and* $H(P_1)$, *respectively. Then,*

(i) $S(P_1) < S(P_2) \Rightarrow P_1 < P_2$
(ii) $S(P_1) = S(P_2)$, *and*
  (a) $H(P_{1.}) < H(P_{2.}) \Rightarrow P_{1.} < P_2$,
  (b) $H(P_{1.}) = H(P_{2.}) \Rightarrow P_1 = P_2$.

PICTURE CUBIC FUZZY HAMACHER AVERAGING AOPs

In this section, first we present some operational laws for PCFS. We introduce a number of PCF Hamacher AOPs and discuss some of their characteristics in this section.

Hamacher operational laws for picture cubic fuzzy set

Let

$$P_1 = \{(v, \langle [\dot{z}_1^-, \dot{z}_1^+], \vartheta_1 \rangle, \langle [\check{s}_1^-, \check{s}_1^+], \delta_1 \rangle, \langle [\dot{c}_1^-, \dot{c}_1^+], \dot{e}_1 \rangle) \mid v \in \exists\}$$

and

$$P_2 = \{(v, \langle [\dot{z}_2^-, \dot{z}_2^+], \vartheta_2 \rangle, \langle [\check{s}_2^-, \check{s}_2^+], \delta_2 \rangle, \langle [\dot{c}_2^-, \dot{c}_2^+], \dot{e}_2 \rangle) \mid v \in \exists\}$$

be PCFS in v, and $k > 0$, we present the following Hamacher operations in PCFS:

(1):

$$P_1 \oplus P_2 = \left\{ \begin{array}{l} \left(\left[\frac{\dot{z}_1^- + \dot{z}_2^- - \dot{z}_1^- \dot{z}_2^- - (1-\gimel)\dot{z}_1^- \dot{z}_2^-}{1-(1-\gimel)\dot{z}_1^- \dot{z}_2^-}, \frac{\dot{z}_1^+ + \dot{z}_2^+ - \dot{z}_1^+ \dot{z}_2^+ - (1-\gimel)\dot{z}_1^+ \dot{z}_2^+}{1-(1-\gimel)\dot{z}_1^+ \dot{z}_2^+} \right], \frac{\vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2 - (1-\gimel)\vartheta_1\vartheta_2}{1-(1-\gimel)\vartheta_1\vartheta_2} \right), \\ \left(\left[\frac{\check{s}_1^- \check{s}_2^-}{\gimel + (1-\gimel)(\check{s}_1^- + \check{s}_2^- - \check{s}_1^- \check{s}_2^-)}, \frac{\check{s}_1^+ \check{s}_2^+}{\gimel + (1-\gimel)(\check{s}_1^+ + \check{s}_2^+ - \check{s}_1^+ \check{s}_2^+)} \right], \frac{\delta_1\delta_2}{\gimel + (1-\gimel)(\delta_1 + \delta_2 - \delta_1\delta_2)} \right), \\ \left(\left[\frac{\dot{c}_1^- \dot{c}_2^-}{\gimel + (1-\gimel)(\dot{c}_1^- + \dot{c}_2^- - \dot{c}_1^- \dot{c}_2^-)}, \frac{\dot{c}_1^+ \dot{c}_2^+}{\gimel + (1-\gimel)(\dot{c}_1^+ + \dot{c}_2^+ - \dot{c}_1^+ \dot{c}_2^+)} \right], \frac{\dot{e}_1\dot{e}_2}{\gimel + (1-\gimel)(\dot{e}_1 + \dot{e}_2 - \dot{e}_1\dot{e}_2)} \right) \end{array} \right\}$$

(2):

$$P_1 \otimes P_2 = \left\{ \begin{array}{l} \left(\left[\frac{\dot{z}_1^- \dot{z}_2^-}{\gimel + (1-\gimel)(\dot{z}_1^- + \dot{z}_2^- - \dot{z}_1^- \dot{z}_2^-)}, \frac{\dot{z}_1^+ \dot{z}_2^+}{\gimel + (1-\gimel)(\dot{z}_1^+ + \dot{z}_2^+ - \dot{z}_1^+ \dot{z}_2^+)} \right], \frac{\vartheta_1\vartheta_2}{\gimel + (1-\gimel)(\vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2)} \right), \\ \left(\left[\frac{\check{s}_1^- + \check{s}_2^- - \check{s}_1^- \check{s}_2^- - (1-\gimel)\check{s}_1^- \check{s}_2^-}{1-(1-\gimel)\check{s}_1^- \check{s}_2^-}, \frac{\check{s}_1^+ + \check{s}_2^+ - \check{s}_1^+ \check{s}_2^+ - (1-\gimel)\check{s}_1^+ \check{s}_2^+}{1-(1-\gimel)\check{s}_1^+ \check{s}_2^+} \right], \frac{\delta_1 + \delta_2 - \delta_1\delta_2 - (1-\gimel)\delta_1\delta_2}{1-(1-\gimel)\delta_1\delta_2} \right), \\ \left(\left[\frac{\dot{c}_1^- + \dot{c}_2^- - \dot{c}_1\dot{c}_2(1-\gimel)\dot{c}_1\dot{c}_2}{1-(1-\gimel)\check{s}_1^- \check{s}_2^-}, \frac{\dot{c}_1^+ + \dot{c}_2^+ - \dot{c}_1^+ \dot{c}_2^+ - (1-\gimel)\dot{c}_1^+ \dot{c}_2^+}{1-(1-\gimel)\check{s}_1^+ \check{s}_2^+} \right], \frac{\dot{e}_1 + \dot{e}_2 - \dot{e}_1\dot{e}_2 - (1-\gimel)\dot{e}_1\dot{e}_2}{1-(1-\gimel)\dot{e}_1\dot{e}_2} \right) \end{array} \right\}$$

(3):

$$\kappa P = \left\{ \begin{array}{l} \left(\left[\frac{(1+(\gimel-1)\dot{z}^-)^\kappa - (1-\dot{z}^-)^\kappa}{(1+(\gimel-1)\dot{z}^-)^\kappa + (\gimel-1)(1-\dot{z}^-)^\kappa}, \frac{(1+(\gimel-1)\dot{z}^+)^\kappa - (1-\dot{z}^+)^\kappa}{(1+(\gimel-1)\dot{z}^+)^\kappa + (\gimel-1)(1-\dot{z}^+)^\kappa} \right], \frac{(1+(\gimel-1)\vartheta)^\kappa - (1-\vartheta)^\kappa}{(1+(\gimel-1)\vartheta)^\kappa + (\gimel-1)(1-\vartheta)^\kappa} \right), \\ \left(\left[\frac{\gimel(\check{s}^-)^\kappa}{1+(\gimel-1)(1-\check{s}^-)^\kappa + (\gimel-1)(\check{s}^-)^\kappa}, \frac{\gimel(\check{s}^+)^\kappa}{1+(\gimel-1)(1-\check{s}^+)^\kappa + (\gimel-1)(\check{s}^+)^\kappa} \right], \frac{\gimel(\delta)^\kappa}{1+(\gimel-1)(1-\delta)^\kappa + (\gimel-1)\delta^\kappa} \right), \\ \left(\left[\frac{\gimel(\dot{c})^\kappa}{1+(\gimel-1)(1-\dot{c}^-)^\kappa + (\gimel-1)(\dot{c}^-)^\kappa}, \frac{\gimel(\dot{c}^+)^\kappa}{1+(\gimel-1)(1-\dot{c}^+)^\kappa + (\gimel-1)(\dot{c}^+)^\kappa} \right], \frac{\gimel(\dot{e})^\kappa}{1+(\gimel-1)(1-\dot{e})^\kappa + (\gimel-1)\dot{e}^\kappa} \right) \end{array} \right\}$$

(4):

$$P^\kappa = \left\{ \begin{array}{l} \left(\left[\frac{\gimel(\dot{z}^-)^\kappa}{(1+(\gimel-1)(1-\dot{z}^-))^\kappa + (\gimel-1)(\dot{z}^-)^\kappa}, \frac{\gimel(\dot{z}^+)^\kappa}{(1+(\gimel-1)(1-\dot{z}^+))^\kappa + (\gimel-1)(\dot{z}^+)^\kappa} \right], \frac{\gimel(\vartheta)^\kappa}{(1+(\gimel-1)(1-\vartheta))^\kappa + (\gimel-1)\vartheta^\kappa} \right), \\ \left(\left[\frac{(1+(\gimel-1)\check{s}^-)^\kappa - (1-\check{s}^-)^\kappa}{(1+(\gimel-1)\check{s}^-)^\kappa + (\gimel-1)(1-\check{s}^-)^\kappa}, \frac{(1+(\gimel-1)\check{s}^+)^\kappa - (1-\check{s}^+)^\kappa}{(1+(\gimel-1)\check{s}^+)^\kappa + (\gimel-1)(1-\check{s}^+)^\kappa} \right], \frac{(1+(\gimel-1)\delta)^\kappa - (1-\delta)^\kappa}{(1+(\gimel-1)\delta)^\kappa + (\gimel-1)(1-\delta)^\kappa} \right), \\ \left(\left[\frac{(1+(\gimel-1)\dot{c}^-)^\kappa - (1-\dot{c}^-)^\kappa}{(1+(\gimel-1)\dot{c}^-)^\kappa + (\gimel-1)(1-\dot{c}^-)^\kappa}, \frac{(1+(\gimel-1)\dot{c}^+)^\kappa - (1-\dot{c}^+)^\kappa}{(1+(\gimel-1)\dot{c}^+)^\kappa + (\gimel-1)(1-\dot{c}^+)^\kappa} \right], \frac{(1+(\gimel-1)\dot{e})^\kappa - (1-\dot{e})^\kappa}{(1+(\gimel-1)\dot{e})^\kappa + (\gimel-1)(1-\dot{e})^\kappa} \right) \end{array} \right\}$$

PCFWA Opt

Definition 5. *Let* $P_\daleth = < c_{P_\daleth}, c'_{P'_\daleth}, c''_{P'_\daleth} > (\daleth = 1, \ldots \ddot{\imath})$ *be a set of PCFVs in* $\exists$ *and let PCFHWA operator of dimension* $\ddot{\imath}$ *be a function* $\Omega^{\ddot{\imath}} \to \Omega$ *and* $\varkappa_\daleth \in [0, 1]$ *such that*

$$PCFHWA_\varkappa(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) = \bigoplus_{\daleth=1}^{\ddot{\imath}} \varkappa_\daleth P_\daleth.$$

Utilizing Hamacher operations on PCFNs, the below theorem is formed.

Theorem 1. *Suppose* $P_{\daleth} =< c_{P_{\daleth}}, c'_{P'_{\daleth}}, c''_{P'_{\daleth}} > (\daleth = 1, \ldots, \ddot{\imath})$, *be PCFNs in* $\exists$, *then using the PCFHWA operator, their aggregated value is also a PCFN and is defined as follows:*

$$
PCFHWA_{\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)
= \left\{ \begin{array}{l}
\left(\left[\begin{array}{l}
\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \\
\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}
\end{array} \right], \right. \\
\left. \qquad \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}} \right), \\
\left(\left[\begin{array}{l}
\dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\breve{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\breve{s}_{\daleth}^{-}))^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\breve{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \\
\dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\breve{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\breve{s}_{\daleth}^{+}))^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\breve{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}
\end{array} \right], \right. \\
\left. \qquad \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\delta_{\daleth}))^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa_{\daleth}}} \right), \\
\left(\left[\begin{array}{l}
\dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{-}))^{\varkappa_{\daleth}} + \left(\gimel-1\right)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \\
\dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{+}))^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}
\end{array} \right], \right. \\
\left. \qquad \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{e}_{\daleth}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{e}_{\daleth}))^{\varkappa_{\daleth}} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}} \right)
\end{array} \right\}, \tag{2}
$$

where $\varkappa = (\varkappa_1, \ldots, \varkappa_{\ddot{\imath}})^T$ *is the associated weight vector (WV) of* $P_{\daleth}$ *with* $\varkappa_{\daleth} > 0$ *and* $\Sigma_{\daleth=1}^{\ddot{\imath}}\varkappa_{\daleth} = 1$.

Proof. We will use mathematical induction to prove this theorem as follows:
when $\daleth = 1$, we have the accompanying result by Hamacher operations on PCFNs,

$$
PCFHWA_{\varkappa}\left(P_1, P_2\right)
= \left\{ \begin{array}{l}
\left(\left[\dfrac{1+(\gimel-1)\dot{z}-(1-\dot{z}^{-})}{1+(\gimel-1)\dot{z}^{-}+(\gimel-1)(1-\dot{z}^{-})}, \dfrac{1+(\gimel-1)\dot{z}^{+}-(1-\dot{z}^{+})}{1+(\gimel-1)\dot{z}^{+}+(\gimel-1)(1-\dot{z}^{+})} \right], \dfrac{1+(\gimel-1)\vartheta-(1-\vartheta)}{1+(\gimel-1)\vartheta+(\gimel-1)(1-\vartheta)} \right), \\
\left(\left[\dfrac{\gimel(\breve{s}^{-})}{1+(\gimel-1)(1-\breve{s}^{-})+(\gimel-1)(\breve{s}^{-})}, \dfrac{\gimel(\breve{s}^{+})}{1+(\gimel-1)(1-\breve{s}^{+})+(\gimel-1)(\breve{s}^{+})} \right], \dfrac{\gimel(\delta)}{1+(\gimel-1)(1-\delta)+(\gimel-1)\delta} \right), \\
\left(\left[\dfrac{\gimel(\dot{c}^{-})}{1+(\gimel-1)(1-\dot{c}^{-})+(\gimel-1)(\dot{c}^{-})}, \dfrac{\gimel(\dot{c}^{+})}{1+(\gimel-1)(1-\dot{c}^{+})+(\gimel-1)(\dot{c}^{+})} \right], \dfrac{\gimel(\dot{e})}{1+(\gimel-1)(1-\dot{e})+(\gimel-1)\dot{e}} \right)
\end{array} \right\}.
$$

Hence (2) is true for $\daleth = 1$.
Let (2) be true for $\daleth = \kappa$, then from 2, we have

$$
\begin{aligned}
&PCFHWA_{\varkappa}\left(P_1,P_2,P_3,\ldots,P_{\kappa}\right)\\
&=\bigoplus_{\daleth=1}^{\kappa}\varkappa_{\daleth}P_{\daleth}\\
&=\left\{\begin{array}{c}
\left(\begin{array}{c}\left[\dfrac{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\kappa}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}},\dfrac{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\kappa}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right],\\
\dfrac{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\kappa}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}\end{array}\right),\\
\left(\begin{array}{c}\left[\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\check{s}_{\daleth}^{-}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}},\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\check{s}_{\daleth}^{+}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right],\\
\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}(\delta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\delta_{\daleth}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\delta_{\daleth})^{\varkappa_{\daleth}}}\end{array}\right),\\
\left(\begin{array}{c}\left[\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{-}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}},\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{+}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right],\\
\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\dot{e}_{\daleth}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}}\end{array}\right)
\end{array}\right\}.
\end{aligned}
$$

Now for $\ddot{\imath}=\kappa+1$, we have

$$
\begin{aligned}
&PCFHWA_{\varkappa}(P_1,P_2,P_3,\ldots,P_{\kappa},P_{\kappa+1})\\
&=\bigoplus_{\daleth=1}^{\kappa}\varkappa_{\daleth}P_{\daleth}\bigoplus\varkappa_{\kappa+1}P_{\kappa+1}\\
&=\left\{\begin{array}{c}
\left(\begin{array}{c}\left[\dfrac{\Pi_{\daleth=1}^{\kappa}\left(1+(\gimel-1)\dot{z}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\kappa}\left(1-\dot{z}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}\left(1+(\gimel-1)\dot{z}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}\left(1-\dot{z}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}},\dfrac{\Pi_{\daleth=1}^{\kappa}\left(1+(\gimel-1)\dot{z}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\kappa}\left(1-\dot{z}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}\left(1+(\gimel-1)\dot{z}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}\left(1-\dot{z}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}}\right],\\
\dfrac{\Pi_{\daleth=1}^{\kappa}\left(1+(\gimel-1)\vartheta_{\daleth}\right)^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\kappa}\left(1-\vartheta_{\daleth}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}\left(1+(\gimel-1)\vartheta_{\daleth}\right)^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}\end{array}\right),\\
\left(\begin{array}{c}\left[\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}\left(\check{s}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\check{s}_{\daleth}^{-}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}},\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}\left(\check{s}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\check{s}_{\daleth}^{+}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right],\\
\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}\left(\delta_{\daleth}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\delta_{\daleth}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\delta_{\daleth})^{\varkappa_{\daleth}}}\end{array}\right),\\
\left(\begin{array}{c}\left[\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}\left(\dot{c}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{-}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}},\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}\left(\dot{c}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{+}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right],\\
\dfrac{\gimel\Pi_{\daleth=1}^{\kappa}\left(\dot{e}_{\daleth}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\kappa}(1+(\gimel-1)(1-\dot{e}_{\daleth}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\kappa}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}}\end{array}\right)
\end{array}\right\}
\end{aligned}
$$

$$
\oplus\left\{\begin{array}{c}
\left(\left[\dfrac{\beth\left(\dot{c}^{-}_{\kappa+1}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\dot{c}^{-}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)(\dot{c}^{-}_{\kappa+1})^{\kappa+1}},\dfrac{\beth\left(\dot{c}^{+}_{\kappa+1}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\dot{c}^{+}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)(\dot{c}^{+}_{\kappa+1})^{\kappa+1}}\right],\right.\\
\left.\dfrac{\beth\left(\dot{e}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\dot{e}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)\dot{e}^{\kappa+1}_{\kappa+1}}\right),\\
\left(\left[\dfrac{\beth\left(\check{s}^{-}_{\kappa+1}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\check{s}^{-}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)(\check{s}^{-}_{\kappa+1})^{\kappa+1}},\dfrac{\beth\left(\check{s}^{+}_{\kappa+1}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\check{s}^{+}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)(\check{s}^{+}_{\kappa+1})^{\kappa+1}}\right],\right.\\
\left.\dfrac{\beth\left(\delta\right)^{\kappa+1}}{1+(\beth-1)\left(1-\delta_{\kappa+1}\right)^{\kappa+1}+(\beth-1)\delta^{\kappa+1}_{\kappa+1}}\right),\\
\left(\left[\dfrac{\beth\left(\dot{z}^{-}_{\kappa+1}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\dot{z}^{-}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)(\dot{z}^{-}_{\kappa+1})^{\kappa+1}},\dfrac{\beth\left(\dot{z}^{+}_{\kappa+1}\right)^{\kappa+1}}{1+(\beth-1)\left(1-\dot{z}^{+}_{\kappa+1}\right)^{\kappa+1}+(\beth-1)(\dot{z}^{+}_{\kappa+1})^{\kappa+1}}\right],\right.\\
\left.\dfrac{\beth\left(\vartheta\right)^{\kappa+1}}{1+(\beth-1)\left(1-\vartheta_{\kappa+1}\right)^{\kappa+1}+(\beth-1)\vartheta^{\kappa+1}_{\kappa+1}}\right)
\end{array}\right\}
$$

$$
=\left\{\begin{array}{c}
\left(\left[\dfrac{\Pi_{\daleth=1}^{\kappa+1}\left(1+(\beth-1)\dot{z}^{-}_{\daleth}\right)^{\varkappa_\daleth}-\Pi_{\daleth=1}^{\kappa+1}\left(1-\dot{z}^{-}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}\left(1+(\beth-1)\dot{z}^{-}_{\daleth}\right)^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}\left(1-\dot{z}^{-}_{\daleth}\right)^{\varkappa_\daleth}},\dfrac{\Pi_{\daleth=1}^{\kappa+1}\left(1+(\beth-1)\dot{z}^{+}_{\daleth}\right)^{\varkappa_\daleth}-\Pi_{\daleth=1}^{\kappa+1}\left(1-\dot{z}^{+}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}\left(1+(\beth-1)\dot{z}^{+}_{\daleth}\right)^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}\left(1-\dot{z}^{+}_{\daleth}\right)^{\varkappa_\daleth}}\right],\right.\\
\left.\dfrac{\Pi_{\daleth=1}^{\kappa+1}\left(1+(\beth-1)\vartheta_{\daleth}\right)^{\varkappa_\daleth}-\Pi_{\daleth=1}^{\kappa+1}\left(1-\vartheta_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}\left(1+(\beth-1)\vartheta_{\daleth}\right)^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(1-\vartheta_{\daleth})^{\varkappa_\daleth}}\right),\\
\left(\left[\dfrac{\beth\Pi_{\daleth=1}^{\kappa+1}\left(\check{s}^{-}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}(1+(\beth-1)\left(1-\check{s}^{-}_{\daleth}\right))^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(\check{s}^{-}_{\daleth})^{\varkappa_\daleth}},\dfrac{\beth\Pi_{\daleth=1}^{\kappa+1}\left(\check{s}^{+}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}(1+(\beth-1)\left(1-\check{s}^{+}_{\daleth}\right))^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(\check{s}^{+}_{\daleth})^{\varkappa_\daleth}}\right],\right.\\
\left.\dfrac{\beth\Pi_{\daleth=1}^{\kappa+1}\left(\delta_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}(1+(\beth-1)(1-\delta_{\daleth}))^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(\delta_{\daleth})^{\varkappa_\daleth}}\right),\\
\left(\left[\dfrac{\beth\Pi_{\daleth=1}^{\kappa+1}\left(\dot{c}^{-}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}(1+(\beth-1)(1-\dot{c}^{-}_{\daleth}))^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(\dot{c}^{-}_{\daleth})^{\varkappa_\daleth}},\dfrac{\beth\Pi_{\daleth=1}^{\kappa+1}\left(\dot{c}^{+}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}(1+(\beth-1)(1-\dot{c}^{+}_{\daleth}))^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(\dot{c}^{+}_{\daleth})^{\varkappa_\daleth}}\right],\right.\\
\left.\dfrac{\beth\Pi_{\daleth=1}^{\kappa+1}\left(\dot{e}_{\daleth}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\kappa+1}(1+(\beth-1)(1-\dot{e}_{\daleth}))^{\varkappa_\daleth}+(\beth-1)\Pi_{\daleth=1}^{\kappa+1}(\dot{e}_{\daleth})^{\varkappa_\daleth}}\right),
\end{array}\right\}
$$

Thus (2) is true for $\ddot{\imath}=\kappa+1$. Thus through mathematical induction (2) is valid for all values of $\daleth$. □

Sensitivity analysis

In this section, we show the sensitivity analysis with respect to the parameter $\beth$. We have two cases for PCFHWA Opt with respect to the change in the value of parameter $\beth$.

Case 1: When $\beth = 1$, the structure of PCFHWA Opt will obtain the form of PCFWA Opt.

$$
PCFWA_{\varkappa}\left(P_1,P_2,P_3,\ldots,P_{\ddot{\imath}}\right)
=\left\{\begin{array}{c}
([1-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{-}_{\daleth})^{\varkappa_\daleth},),1-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{+}_{\daleth})^{\varkappa_\daleth}],1-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_\daleth},\\
([\Pi_{\daleth=1}^{\ddot{\imath}}\check{s}^{-\,\varkappa_\daleth}_{\daleth},\Pi_{\daleth=1}^{\ddot{\imath}}\check{s}^{+\,\varkappa_\daleth}_{\daleth}],\Pi_{\daleth=1}^{\ddot{\imath}}\delta^{\varkappa_\daleth}_{\daleth}),\\
([\Pi_{\daleth=1}^{\ddot{\imath}}\dot{c}^{-\,\varkappa_\daleth}_{\daleth},\Pi_{\daleth=1}^{\ddot{\imath}}\dot{c}^{+\,\varkappa_\daleth}_{\daleth}],\Pi_{\daleth=1}^{\ddot{\imath}}\dot{e}^{\varkappa_\daleth}_{\daleth})
\end{array}\right\}, \tag{3}
$$

Case 2: When $\beth = 2$, the structure of PCFHWA Opt will obtain the form of PCF Einstein weighted averaging Opt.

$$PCFEWA_{\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)$$

$$= \left\{ \begin{array}{l} \left(\left[\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right], \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\vartheta_{\daleth})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\vartheta_{\daleth})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}\right), \\ \left(\left[\frac{2\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(2-\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \frac{2\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(2-\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right], \frac{2\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\delta_{\daleth}\right)^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa_{\daleth}}}\right), \\ \left(\left[\frac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\daleth}^{-}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(2-\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \frac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\daleth}^{+}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(2-\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right], \frac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{e}_{\daleth}\right)^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\dot{e}_{\daleth}\right)^{\varkappa_{\daleth}} + \Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}}\right) \end{array} \right\}. \quad (4)$$

Based on Theorem 1 we have the below properties of PCFHWA Opt.

Proposition 1: *Let* $P_{\daleth} =< c_{P_{\daleth}}, c'_{P'_{\daleth}}, c''_{P'_{\daleth}} >, (\daleth = 1, 2, \ldots, \ddot{\imath})$ *be PCFNs in G and* $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_{\ddot{\imath}})^T$*, the WV of* $P_{\daleth}$ *with* $\Sigma_{\daleth=1}^{\ddot{\imath}}\varkappa_{\daleth} = 1$ *and* $\varkappa_{\daleth} > 0$*, then the following properties are formed.*

Boundedness property: For every $\varkappa$,

$$P^{-} \leq PCFHWG_{\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) \leq P^{+}$$

where

$$P^{+} = \{([\min \dot{z}_{\daleth}^{-}, \max \dot{z}_{\daleth}^{+}], \max \vartheta_{\daleth}), ([\max \check{s}_{\daleth}^{-}, \min \check{s}_{\daleth}^{+}], \min \delta_{\daleth}), ([\max \dot{c}_{\daleth}^{-}, \min \dot{c}_{\daleth}^{+}], \min \dot{e}_{\daleth})\}$$

$$P^{-} = \{([\max \dot{z}_{\daleth}^{-}, \min \dot{z}_{\daleth}^{+}], \min \vartheta_{\daleth}), ([\min \check{s}_{\daleth}^{-}, \max \check{s}_{\daleth}^{+}], \max \delta_{\daleth}), ([\min \dot{c}_{\daleth}^{-}, \max \dot{c}_{\daleth}^{+}], \max \dot{e}_{\daleth})\}.$$

Idempotency property: If all $P_{\daleth} =< c_{P_{\daleth}}, c'_{P'_{\daleth}}, c''_{P'_{\daleth}} >, (\daleth = 1, 2, \ldots, \ddot{\imath})$ are equal, *P.e.*, $P_{\daleth} = P$, then $PCFHWA_{\kappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) = P$

Proof. Since for all $\daleth$, $P_{\daleth} = P$, *P.e.*, $\dot{z}_{\daleth}^{-} = \dot{z}^{-}$, $\dot{z}_{\daleth}^{+} = \dot{z}^{+}$, $\check{s}_{\daleth}^{-} = \check{s}^{-}$, $\check{s}_{\daleth}^{+} = \check{s}^{+}$,

$$PCFHWA_{\varkappa}\left(P_1, P_2, \ldots, P_{\ddot{\imath}}\right)$$

$$= \left\{ \begin{array}{l} \left(\begin{array}{c} \left[\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right], \\ \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+\left(\beth-1\right)\vartheta_{\daleth})^{\varkappa_{\daleth}} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)\vartheta_{\daleth})^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa_{\daleth}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\frac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)(1-\check{s}_{\daleth}^{-}))^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \frac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)(1-\check{s}_{\daleth}^{+}))^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right], \\ \frac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)(1-\delta_{\daleth}))^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa_{\daleth}}} \end{array} \right), \\ \left(\begin{array}{c} \left[\frac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)(1-\dot{c}_{\daleth}^{-}))^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{-})^{\varkappa_{\daleth}}}, \frac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)(1-\dot{c}_{\daleth}^{+}))^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{+})^{\varkappa_{\daleth}}}\right], \\ \frac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\beth-1)(1-\dot{e}_{\daleth}))^{\varkappa_{\daleth}} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth})^{\varkappa_{\daleth}}} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left(\left[\frac{1+(\gimel-1)\dot{z}^- - (1-\dot{z}^-)}{(1+(\gimel-1)\dot{z}^-)+(\gimel-1)(1-\dot{z}^-)}, \frac{1+(\gimel-1)\dot{z}^+ - (1-\dot{z}^+)}{(1+(\gimel-1)\dot{z}^+)+(\gimel-1)(1-\dot{z}^+)} \right], \frac{1+(\gimel-1)\vartheta-(1-\vartheta)}{1+(\gimel-1)\vartheta+(\gimel-1)(1-\vartheta)} \right), \\ \left(\left[\frac{\gimel(\check{s}^-)}{1+(\gimel-1)(1-\check{s}^-)+(\gimel-1)(\check{s}^-)}, \frac{\gimel(\check{s}^+)}{1+(\gimel-1)(1-\check{s}^+)+(\gimel-1)(\check{s}^+)} \right], \frac{\gimel(\delta)}{1+(\gimel-1)(1-\delta)+(\gimel-1)\delta} \right), \\ \left(\left[\frac{\gimel(\dot{c}^-)}{1+(\gimel-1)(1-\dot{c}^-)+(\gimel-1)(\dot{c}^-)}, \frac{\gimel(\dot{c}^+)}{1+(\gimel-1)(1-\dot{c}^+)+(\gimel-1)(\dot{c}^+)} \right], \frac{\gimel(\dot{e})}{1+(\gimel-1)(1-\dot{e})+(\gimel-1)\dot{e}} \right) \end{array} \right\}$$

$$= \{([\dot{z}^-, \dot{z}^+], \vartheta), ([\check{s}^-, \check{s}^+], \delta), ([\partial^-, \partial^+], \dot{e})\}$$

$$= P,$$

proved. □

Monotonicity property: Let $P^* = \{([\dot{z}_\daleth^{*-}, \dot{z}_\daleth^{*+}], \vartheta_\daleth^*), ([\check{s}_\daleth^{*-}, \check{s}_\daleth^{*+}], \delta_\daleth^*), ([\dot{c}_\daleth^{*-}, \dot{c}_\daleth^{*+}], \dot{e}_\daleth^*)\}$, ($\daleth = 1, 2, \ldots, \ddot{\imath}$) be a group of PCFVs, if $[\dot{z}_\daleth^-, \dot{z}_\daleth^+] \le [\dot{z}_\daleth^{*-}, \dot{z}_\daleth^{*+}]$, $\vartheta_\daleth \le \vartheta_\daleth^*$, $[\check{s}_\daleth^{*-}, \check{s}_\daleth^{*+}] \le [\check{s}_\daleth^-, \check{s}_\daleth^+]$, $\delta_\daleth^* \le \delta_\daleth$, $[\dot{c}_\daleth^{*-}, \dot{c}_\daleth^{*+}] \le [\dot{c}_\daleth^-, \dot{c}_\daleth^+]$, $\dot{e}_\daleth^* \le \dot{e}_\daleth$ for all $\daleth$. Then

$$PCFHWA_\varkappa(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) \le PCFHWA_\varkappa(P_1^*, P_2^*, P_3^*, \ldots, P_{\ddot{\imath}}^*).$$

Proof.

$$PCFHWA_\varkappa(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}})$$

$$= \left\{ \begin{array}{l} \left(\begin{array}{c} \left[\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^-)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^-)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^-)^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^-)^{\varkappa\daleth}}, \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^+)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^+)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^+)^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^+)^{\varkappa\daleth}} \right], \\ \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_\daleth)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_\daleth)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_\daleth)^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_\daleth)^{\varkappa\daleth}} \end{array} \right), \\ \left(\begin{array}{c} \left[\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^-)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}_\daleth^-))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^-)^{\varkappa\daleth}}, \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^+)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}_\daleth^+))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^+)^{\varkappa\daleth}} \right], \\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_\daleth)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\delta_\daleth))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_\daleth)^{\varkappa\daleth}} \end{array} \right), \\ \left(\begin{array}{c} \left[\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_\daleth^-)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}_\daleth^-))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_\daleth^-)^{\varkappa\daleth}}, \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_\daleth^+)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}_\daleth^+))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_\daleth^+)^{\varkappa\daleth}} \right], \\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_\daleth)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{e}_\daleth))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_\daleth)^{\varkappa\daleth}} \end{array} \right) \end{array} \right\}, \quad (5)$$

as

$$\Rightarrow \left[\begin{array}{c} \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^{*-})^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^{*-})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^{*-})^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^{*-})^{\varkappa\daleth}}, \\ \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^{*+})^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^{*+})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^{*+})^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^{*+})^{\varkappa\daleth}} \end{array} \right] \ge \left[\begin{array}{c} \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^-)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^-)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^-)^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^-)^{\varkappa\daleth}}, \\ \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^+)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^+)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_\daleth^+)^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_\daleth^+)^{\varkappa\daleth}} \end{array} \right] \quad (6)$$

now since,

$$\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^{*-})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}_\daleth^{*-}))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^{*-})^{\varkappa\daleth}}$$

$$\le \quad \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^-)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}_\daleth^-))^{\varkappa\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_\daleth^-)^{\varkappa\daleth}},$$

$$\begin{aligned}&\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{*+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{*+}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{*+}_{\daleth})^{\varkappa\daleth}}\\ \geq\ &\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{+}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{+}_{\daleth})^{\varkappa\daleth}}\end{aligned}$$

$$\begin{bmatrix}\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{*-}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{*-}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{*-}_{\daleth})^{\varkappa\daleth}},\\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{*+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{*+}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{*+}_{\daleth})^{\varkappa\daleth}}\end{bmatrix}\leq\begin{bmatrix}\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{-}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{-}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{-}_{\daleth})^{\varkappa\daleth}},\\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{+}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{+}_{\daleth})^{\varkappa\daleth}}\end{bmatrix} \tag{7}$$

since

$$\begin{aligned}&\vartheta_{\daleth}\geq\vartheta^{*}_{\daleth}\\ &\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa\daleth}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa\daleth}}\\ \geq\ &\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta^{*}_{\daleth})^{\varkappa\daleth}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta^{*}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta^{*}_{\daleth})^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta^{*}_{\daleth})^{\varkappa\daleth}}\end{aligned} \tag{8}$$

and since,

$$\begin{aligned}&\delta^{*}_{\daleth}\geq\delta_{\daleth}\\ &\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\delta^{*}_{\daleth}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\left(1-\delta^{*}_{\daleth}\right))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta^{*}_{\daleth})^{\varkappa\daleth}}\\ \geq\ &\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\delta_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa\daleth}}\end{aligned} \tag{9}$$

equation 6 to 9 imply

$$\left\{\begin{matrix}\left(\begin{matrix}\left[\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}^{-}_{\daleth})^{\varkappa\daleth}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{-}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}^{-}_{\daleth})^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{-}_{\daleth})^{\varkappa\daleth}},\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}^{+}_{\daleth})^{\varkappa\daleth}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}^{+}_{\daleth})^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{+}_{\daleth})^{\varkappa\daleth}}\right],\\ \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa\daleth}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth})^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth})^{\varkappa\daleth}}\end{matrix}\right),\\ \left(\begin{matrix}\left[\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{-}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{-}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{-}_{\daleth})^{\varkappa\daleth}},\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}^{+}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}^{+}_{\daleth})^{\varkappa\daleth}}\right],\\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\delta_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth})^{\varkappa\daleth}}\end{matrix}\right),\\ \left(\begin{matrix}\left[\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}^{-}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}^{-}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}^{-}_{\daleth})^{\varkappa\daleth}},\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}^{+}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}^{+}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}^{+}_{\daleth})^{\varkappa\daleth}}\right],\\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth})^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{e}_{\daleth}))^{\varkappa\daleth}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth})^{\varkappa\daleth}}\end{matrix}\right)\end{matrix}\right\}$$

$$\leq\left\{\begin{array}{c}\left(\begin{array}{c}\left[\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}},\frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\dot{z}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}}\right],\\ \frac{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth}^{*})^{\varkappa_{\daleth}}-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth}^{*})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)\vartheta_{\daleth}^{*})^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_{\daleth}^{*})^{\varkappa_{\daleth}}}\end{array}\right),\\ \left(\begin{array}{c}\left[\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}_{\daleth}^{*^{-}}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}},\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\check{s}_{\daleth}^{*^{+}}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\check{s}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}}\right],\\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth}^{*})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\delta_{\daleth}^{*}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\delta_{\daleth}^{*})^{\varkappa_{\daleth}}}\end{array}\right),\\ \left(\begin{array}{c}\left[\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{*^{-}}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{*^{-}})^{\varkappa_{\daleth}}},\frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{c}_{\daleth}^{*^{+}}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{c}_{\daleth}^{*^{+}})^{\varkappa_{\daleth}}}\right],\\ \frac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth}^{*})^{\varkappa_{\daleth}}}{\Pi_{\daleth=1}^{\ddot{\imath}}(1+(\gimel-1)(1-\dot{e}_{\daleth}^{*}))^{\varkappa_{\daleth}}+(\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}(\dot{e}_{\daleth}^{*})^{\varkappa_{\daleth}}}\end{array}\right)\end{array}\right\}$$

i.e.

$$PCFHWA_{\varkappa}(P_1, P_2, P_3, ..., P_{\ddot{\imath}}) \leq PCFHWA_{\varkappa}(P_1^*, P_2^*, P_3^*, ..., P_{\ddot{\imath}}^*)$$

proved. □

PCFHOWA Opt

Here, we have introduced PCFHOWA Opt and discuss its basic characteristics, i.e. monotonicity, boundedness, and idempotency properties.

Definition 6. *Let* $P_{\daleth} =< c_{P_{\daleth}}, c'_{P'_{\daleth}}, c''_{P'_{\daleth}} > (\daleth = 1, \ldots \ddot{\imath})$, *be PCFVs in* $\exists$. *A PCFHOWA operator of dimension* $\ddot{\imath}$ *is a mapping* $PCFHOWA : \Omega^{\ddot{\imath}} \to \Omega$, *with the WV* $\varkappa = (\varkappa_1, \ldots, \varkappa_{\ddot{\imath}})^T$, *with* $\Sigma_{\daleth=1}^{\ddot{\imath}} \varkappa_{\daleth} = 1$ *and* $\varkappa_{\daleth} > 0$, *as*

$$PCFHOWA_{\varkappa}(P_1, P_2, P_3, ..., P_{\ddot{\imath}}) = \bigoplus_{\daleth=1}^{\ddot{\imath}} \varkappa_{\daleth} P_{\sigma_{(\daleth)}},$$

where for all $\daleth$, $P_{\sigma_{(\daleth-1)}} \geq P_{\sigma_{(\daleth)}}$ *and* $(\sigma_{(1)}, \sigma_{(2)}, \ldots, \sigma_{(\ddot{\imath})})$ *is a permutation of* $(1, 2, \ldots, \ddot{\imath})$. *Utilizing Hamacher operations on PCFNs, the below theorem is formed.*

Theorem 2. *Suppose* $P_{\daleth} =< c_{P_{\daleth}}, c'_{P'_{\daleth}}, c''_{P'_{\daleth}} > (\daleth = 1, \ldots \ddot{\imath})$, *be PCFNs in* $\exists$, *then using the PCFHOWA operator, their aggregated value is also a PCFN and is defined as follows:*

$$PCFHOWA_{\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)$$

$$= \left\{ \begin{array}{c} \left(\left[\begin{array}{c} \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\dot{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\dot{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}, \\ \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\dot{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\dot{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \end{array} \right], \\ \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \right), \\ \left(\left[\begin{array}{c} \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\left(1-\check{s}^{-}_{\sigma(\daleth)}\right)\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}, \\ \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\left(1-\check{s}^{+}_{\sigma(\daleth)}\right)\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \end{array} \right], \\ \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\delta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\left(1-\delta_{\sigma(\daleth)}\right)\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\delta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \right), \\ \left(\left[\begin{array}{c} \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\left(1-\dot{c}^{-}_{\sigma(\daleth)}\right)\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}^{-}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}, \\ \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\left(1-\dot{c}^{+}_{\sigma(\daleth)}\right)\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}^{+}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \end{array} \right], \\ \dfrac{\gimel\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{e}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\gimel-1)\left(1-\dot{e}_{\sigma(\daleth)}\right)\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{e}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \right) \end{array} \right\} \quad (10)$$

where the WV of $P_\daleth$ *is* $\varkappa = (\varkappa_1, \ldots, \varkappa_{\ddot{\imath}})^T$, *with* $\Sigma_{\daleth=1}^{\ddot{\imath}}\varkappa_\daleth = 1$ *and* $\varkappa_\daleth \in [0, 1]$.

Sensitivity analysis

In this section, we show the sensitivity analysis with respect to the parameter $\gimel$. We have two cases for PCFHOWA Opt with respect to the change in the value of parameter $\gimel$.

Case 1: If $\gimel = 1$, then PCFHOWA Opt will obtain the form of picture cubic fuzzy Hamacher order weighted averaging (PCFOWA) Opt,

$$PCFWA_{\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)$$

$$= \left\{ \begin{array}{c} ([1-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{-}_{\daleth})^{\varkappa_\daleth},), 1-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\dot{z}^{+}_{\daleth})^{\varkappa_\daleth}], 1-\Pi_{\daleth=1}^{\ddot{\imath}}(1-\vartheta_\daleth)^{\varkappa_\daleth}, \\ ([\Pi_{\daleth=1}^{\ddot{\imath}}\check{s}_{\daleth}^{-\varkappa_\daleth}, \Pi_{\daleth=1}^{\ddot{\imath}}\check{s}_{\daleth}^{+\varkappa_\daleth}], \Pi_{\daleth=1}^{\ddot{\imath}}\delta_{\daleth}^{\varkappa_\daleth}), \\ ([\Pi_{\daleth=1}^{\ddot{\imath}}\dot{c}_{\daleth}^{-\varkappa_\daleth}, \Pi_{\daleth=1}^{\ddot{\imath}}\dot{c}_{\daleth}^{+\varkappa_\daleth}], \Pi_{\daleth=1}^{\ddot{\imath}}\dot{e}_{\daleth}^{\varkappa_\daleth}) \end{array} \right\}, \quad (11)$$

Case 2: If $\beth = 2$, then PCFHOWA Opt will obtain the form of PCF Einstein order weighted averaging Opt,

$$PCFEOWA_{\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)$$

$$= \left\{ \begin{array}{c} \left(\left[\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\dot{z}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\dot{z}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}, \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\dot{z}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\dot{z}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\dot{z}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}} \right], \\ \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\vartheta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \right), \\ \left(\left[\dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\check{s}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}, \dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\check{s}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\check{s}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}} \right], \\ \dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\delta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\delta_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\delta_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \right), \\ \left(\left[\dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\dot{c}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}, \dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\dot{c}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{c}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}} \right], \\ \dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{e}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\dot{e}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + (\gimel-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\dot{e}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}} \right) \end{array} \right\} \tag{12}$$

Proposition 2. *Let* $P_\daleth = < c_{P_\daleth}, c'_{P'_\daleth}, c''_{P'_\daleth} >, (\daleth = 1, 2, \ldots, \ddot{\imath})$ *be a group of PCFVs in G and the WV of* $P_\daleth$ *be* $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_{\ddot{\imath}})^T$, *with* $\Sigma_{\daleth=1}^{\ddot{\imath}}\varkappa_\daleth = 1$ *and* $\varkappa_\daleth \in [0, 1]$, *then the below characteristics are formed.*

Boundedness property: for every $\varkappa$,

$$P^- \le PCFHOWG_{\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) \le P^+,$$

where

$$P^+ = \left\{ \begin{array}{c} \left(\left[\min \dot{z}_{\sigma(\daleth)}^{-}, \max \dot{z}_{\sigma(\daleth)}^{+}\right], \max \vartheta_{\sigma(\daleth)}\right), \left(\left[\max \check{s}_{\sigma(\daleth)}^{-}, \min \check{s}_{\sigma(\daleth)}^{+}\right], \min \delta_{\sigma(\daleth)}\right), \\ \left(\left[\max \dot{c}_{\sigma(\daleth)}^{-}, \min \dot{c}_{\sigma(\daleth)}^{+}\right], \min \dot{e}_{\sigma(\daleth)}\right) \end{array} \right\},$$

$$P^- = \left\{ \begin{array}{c} \left(\left[\max \dot{z}_{\sigma(\daleth)}^{-}, \min \dot{z}_{\sigma(\daleth)}^{+}\right], \min \vartheta_{\sigma(\daleth)}\right), \left(\left[\min \check{s}_{\sigma(\daleth)}^{-}, \max \check{s}_{\sigma(\daleth)}^{+}\right], \max \delta_{\sigma(\daleth)}\right), \\ \left(\left[\min \dot{c}_{\sigma(\daleth)}^{-}, \max \dot{c}_{\sigma(\daleth)}^{+}\right], \max \dot{e}_{\sigma(\daleth)}\right) \end{array} \right\}.$$

Idempotency property: If all $P_\daleth = < c_{P_\daleth}, c'_{P'_\daleth}, c''_{P'_\daleth} >$ are equal, *i.e.* $P_\daleth = P$, then

$$PCFHOWA_{\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) = P$$

Monotonicity property: Let

$$P^* = \left\{\left(\left[\dot{z}_{\sigma(\daleth)}^{*-}, \dot{z}_{\sigma(\daleth)}^{*+}\right], \vartheta_{\sigma(\daleth)}^{*}\right), \left(\left[\check{s}_{\sigma(\daleth)}^{*-}, \check{s}_{\sigma(\daleth)}^{*+}\right], \delta_{\sigma(\daleth)}^{*}\right), \left(\left[\dot{c}_{\sigma(\daleth)}^{*-}, \dot{c}_{\sigma(\daleth)}^{*+}\right], \dot{e}_{\sigma(\daleth)}^{*}\right)\right\} (\daleth = 1, 2, \ldots, \ddot{\imath})$$

be a group of picture cubic fuzzy values, if

$$\left[\dot{z}_{\sigma(\daleth)}^{-}, \dot{z}_{\sigma(\daleth)}^{+}\right] \le \left[\dot{z}_{\sigma(\daleth)}^{*-}, \dot{z}_{\sigma(\daleth)}^{*+}\right], \vartheta_{\sigma(\daleth)} \le \vartheta_{\sigma(\daleth)}^{*}, \left[\check{s}_{\sigma(\daleth)}^{*-}, \check{s}_{\sigma(\daleth)}^{*+}\right] \le \left[\check{s}_{\sigma(\daleth)}^{-}, \check{s}_{\sigma(\daleth)}^{+}\right], \delta_{\sigma(\daleth)}^{*} \le \delta_{\sigma(\daleth)},$$

$$\left[\dot{c}^{*-}_{\sigma_{(\daleth)}}, \dot{c}^{*+}_{\sigma_{(\daleth)}}\right] \leq \left[\dot{c}^{-}_{\sigma_{(\daleth)}}, \dot{c}^{+}_{\sigma_{(\daleth)}}\right], \dot{e}^{*}_{\sigma_{(\daleth)}} \leq \dot{e}_{\sigma_{(\daleth)}},$$

for all $\daleth$,

$$PCFHOWA_{\varkappa}(P_1, P_2, P_3, ..., P_{\ddot{\imath}}) \leq PCFHOWA_{\varkappa}(P_1^*, P_2^*, P_3^*, ..., P_{\ddot{\imath}}^*).$$

Picture cubic fuzzy Hamacher hybrid averaging Opt

In this section, we present PCFHHA Opt and discuss its basic characteristics.

Definition 7. *Let* $P_\daleth = < c_{P_\daleth}, c'_{P'_\daleth}, c''_{P'_\daleth} >, (\daleth = 1, 2, \ldots, \ddot{\imath})$ *be a set of PCFVs in* $\exists$. *A picture cubic fuzzy Hamacher hybrid averaging (PCFHHA) operator of dimension* $\ddot{\imath}$ *is a function* $PCFHHA : \Omega^{\ddot{\imath}} \to \Omega$, *having an associated WV* $\varkappa = (\varkappa_1, \ldots, \varkappa_{\ddot{\imath}})^T$ *with* $\varkappa_\daleth > 0$ *and* $\Sigma_{\daleth=1}^{\ddot{\imath}} \varkappa_\daleth = 1$. *such that,*

$$PCFHHA_{w,\varkappa}(P_1, P_2, ..., P_{\ddot{\imath}}) = \varkappa_1 \tilde{P}_{\sigma_{(1)}} \oplus \varkappa_2 \tilde{P}_{\sigma_{(2)}} \oplus, ..., \oplus \varkappa_{\ddot{\imath}} \tilde{P}_{\sigma_{(\ddot{\imath})}}.$$

Here $\tilde{P}_{\sigma_{(\daleth)}}$ *is the pth largest of the weighted PCFVs* $\tilde{I}_\daleth$. *Also the WV of* $P_\daleth$ *is* $w = (w_1, \ldots, w_{\ddot{\imath}})^T$, *with* $w_\daleth \in [0, 1]$, *and* $\Sigma_{\daleth=1}^{\ddot{\imath}} w_\daleth = 1$,

$$i.e.\ \tilde{P}_{\sigma_{(\daleth)}} = \ddot{\imath} w_\daleth P_{\sigma_{(\daleth)}} = \left(\left(\left[\tilde{\dot{z}}^{-}_{\sigma_{(\daleth)}}, \tilde{\dot{z}}^{+}_{\sigma_{(\daleth)}}\right], \tilde{\vartheta}_{\sigma_{(\daleth)}}\right), \left(\left[\tilde{\check{s}}^{-}_{\sigma_{(\daleth)}}, \tilde{\check{s}}^{+}_{\sigma_{(\daleth)}}\right], \tilde{\delta}_{\sigma_{(\daleth)}}\right), \left(\left[\tilde{\dot{c}}^{-}_{\sigma_{(\daleth)}}, \tilde{\dot{c}}^{+}_{\sigma_{(\daleth)}}\right], \tilde{\dot{e}}_{\sigma_{(\daleth)}}\right)\right)(\daleth = 1, 2, \ldots, \ddot{\imath}).$$

Where
$$\tilde{\dot{z}}^{-}_{\sigma_{(\daleth)}} = \frac{\left(1+(\beth-1)\dot{z}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} - \left(1-\dot{z}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{\left(1+(\beth-1)\dot{z}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(1-\dot{z}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}, \quad \tilde{\dot{z}}^{+}_{\sigma_{(\daleth)}} = \frac{\left(1+(\beth-1)\dot{z}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} - \left(1-\dot{z}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{\left(1+(\beth-1)\dot{z}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(1-\dot{z}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}},$$

$$\tilde{\vartheta}_{\sigma_{(\daleth)}} = \frac{\left(1+(\beth-1)\vartheta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} - \left(1-\vartheta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{\left(1+(\beth-1)\vartheta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(1-\vartheta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}, \quad \tilde{\check{s}}^{-}_{\sigma_{(\daleth)}} = \frac{\beth\left(\check{s}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{1+(\beth-1)\left(1-\check{s}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(\check{s}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}},$$

$$\tilde{\check{s}}^{+}_{\sigma_{(\daleth)}} = \frac{\beth\left(\check{s}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{1+(\beth-1)\left(1-\check{s}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(\check{s}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}, \quad \tilde{\delta}_{\sigma_{(\daleth)}} = \frac{\beth\left(\delta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{1+(\beth-1)\left(1-\delta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(\delta_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}$$

$$\tilde{\dot{c}}^{-}_{\sigma_{(\daleth)}} = \frac{\beth\left(\dot{c}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{1+(\beth-1)\left(1-\dot{c}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(\dot{c}^{-}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}, \quad \tilde{\dot{c}}^{+}_{\sigma_{(\daleth)}} = \frac{\beth\left(\dot{c}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{1+(\beth-1)\left(1-\dot{c}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(\dot{c}^{+}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}},$$

$$\tilde{\dot{e}}_{\sigma_{(\daleth)}} = \frac{\beth\left(\dot{e}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}{1+(\beth-1)\left(1-\dot{e}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth} + (\beth-1)\left(\dot{e}_{\sigma_{(\daleth)}}\right)^{\ddot{\imath}w_\daleth}}$$

Here $\ddot{\imath}$ is the balancing coefficient, which maintains the balance especially when $\dot{c} = (1/_{\ddot{\imath}}, 1/_{\ddot{\imath}}, 1/_{\ddot{\imath}}, \ldots 1/_{\ddot{\imath}})^T$ then PCFHWA and PCFHOWA AOPs are regarded as a special case of PCFHHA Opt.

Therefore, we obtain the below theorem that results from applying Hamacher operations to PCFVs.

Theorem 3. *Suppose* $P_{\daleth} =< c_{P_{\daleth}}, c'_{P'_{\daleth}}, c''_{P'_{\daleth}} > (\daleth = 1, \ldots \ddot{\imath})$, *be PCFNs in* $\exists$, *then their aggregated value using the PCFHHA operator is also a PCFN and is defined as follows:*

$$
PCFHHA_{w,\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)
$$
$$
= \left\{ \begin{array}{c}
\left(\left[\begin{array}{c}
\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\tilde{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\tilde{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}, \\
\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\tilde{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\tilde{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}
\end{array} \right], \right. \\
\left. \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\tilde{e}_{\sigma(\daleth)}\right)^{\varkappa\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{e}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\tilde{e}_{\sigma(\daleth)}\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{e}_{\sigma(\daleth)}\right)^{\varkappa\daleth}} \right), \\
\left(\left[\begin{array}{c}
\dfrac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{s}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\left(1-\tilde{s}^{-}_{\sigma(\daleth)}\right)\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{s}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}, \\
\dfrac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{s}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\left(1-\tilde{s}^{+}_{\sigma(\daleth)}\right)\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{s}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}
\end{array} \right], \right. \\
\left. \dfrac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\delta}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\left(1-\tilde{\delta}_{\sigma(\daleth)}\right)\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\delta}_{\sigma(\daleth)}\right)^{\varkappa\daleth}} \right), \\
\left(\left[\begin{array}{c}
\dfrac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{c}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\left(1-\tilde{c}^{-}_{\sigma(\daleth)}\right)\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{c}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}, \\
\dfrac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{c}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\left(1-\tilde{c}^{+}_{\sigma(\daleth)}\right)\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{c}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}
\end{array} \right], \right. \\
\left. \dfrac{\beth\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{e}_{\sigma(\daleth)}\right)^{\varkappa\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+(\beth-1)\left(1-\tilde{e}_{\sigma(\daleth)}\right)\right)^{\varkappa\daleth} + (\beth-1)\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{e}_{\sigma(\daleth)}\right)^{\varkappa\daleth}} \right)
\end{array} \right\} \tag{13}
$$

$\varkappa = (\varkappa_1, \ldots, \varkappa_{\ddot{\imath}})^T$ *is the WV of* $P_{\daleth}$, $(\daleth = 1, 2, \ldots, \ddot{\imath})$ *with* $\Sigma_{\daleth=1}^{\ddot{\imath}}\varkappa_{\daleth} = 1$ *and* $\varkappa_{\daleth} \in [0, 1]$.

Sensitivity analysis

In this section, we show the sensitivity analysis with respect to the parameter $\beth$. We have two cases for PCFHHA Opt with respect to the change in the value of parameter $\beth$.

Case 1: If $\beth = 1$, then PCFHHA Opt will obtain the form of PCF hybrid averaging PCFHA Opt.

$$
PCFHA_{w,\varkappa}\left(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}\right)
$$
$$
= \left\{ \begin{array}{c}
\left(\left[1-\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{z}^{-}_{\sigma(\daleth)}\right)^{\varkappa\daleth}, 1-\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{z}^{+}_{\sigma(\daleth)}\right)^{\varkappa\daleth} \right], 1-\Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\vartheta}_{\sigma(\daleth)}\right)^{\varkappa\daleth} \right), \\
\left(\left[\Pi_{\daleth=1}^{\ddot{\imath}}\tilde{s}^{-\,\varkappa\daleth}_{\sigma(\daleth)}, \Pi_{\daleth=1}^{\ddot{\imath}}\tilde{s}^{+\,\varkappa\daleth}_{\sigma(\daleth)} \right], \Pi_{\daleth=1}^{\ddot{\imath}}\tilde{\delta}^{\varkappa\daleth}_{\sigma(\daleth)} \right), \\
\left(\left[\Pi_{\daleth=1}^{\ddot{\imath}}\tilde{c}^{-\,\varkappa\daleth}_{\sigma(\daleth)}, \Pi_{\daleth=1}^{\ddot{\imath}}\tilde{c}^{+\,\varkappa\daleth}_{\sigma(\daleth)} \right], \Pi_{\daleth=1}^{\ddot{\imath}}\tilde{e}^{\varkappa\daleth}_{\sigma(\daleth)} \right)
\end{array} \right\}, \tag{14}
$$

Case 2: If $\beth = 2$, then PCFHHA Opt will obtain the form of PCF Einstein hybrid averaging Opt:

$$
\begin{aligned}
& PCFEHA_{w,\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) \\
&= \left\{ \begin{array}{c}
\left(\begin{array}{c}
\left[\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\tilde{\dot{z}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\dot{z}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\tilde{\dot{z}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\dot{z}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}, \dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}} \right], \\
\dfrac{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\tilde{\vartheta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} - \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\vartheta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(1+\tilde{\vartheta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(1-\tilde{\vartheta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}
\end{array} \right), \\
\left(\begin{array}{c}
\left[\dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{s}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\tilde{\dot{s}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{s}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}, \dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{s}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\tilde{\dot{s}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{s}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}} \right], \\
\dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\delta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\tilde{\delta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\delta}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}
\end{array} \right), \\
\left(\begin{array}{c}
\left[\dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{c}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\tilde{\dot{c}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{c}}_{\sigma(\daleth)}^{-}\right)^{\varkappa_\daleth}}, \dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{c}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\tilde{\dot{c}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{c}}_{\sigma(\daleth)}^{+}\right)^{\varkappa_\daleth}} \right], \\
\dfrac{2\Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{e}}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}{\Pi_{\daleth=1}^{\ddot{\imath}}\left(2-\tilde{\dot{e}}_{\sigma(\daleth)}\right)^{\varkappa_\daleth} + \Pi_{\daleth=1}^{\ddot{\imath}}\left(\tilde{\dot{e}}_{\sigma(\daleth)}\right)^{\varkappa_\daleth}}
\end{array} \right)
\end{array} \right\}
\end{aligned}
\tag{15}
$$

Proposition 3. *Let* $P_\daleth = < c_{P_\daleth}, c'_{P'_\daleth}, c''_{P'_\daleth} >, (\daleth = 1, 2, \ldots, \ddot{\imath})$ *be a group of PCFVs in G and* $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_{\ddot{\imath}})^T$ *be the WV of* $P_\daleth$, *with* $\varkappa_\daleth \in [0, 1]$, *and* $\Sigma_{\daleth=1}^{\ddot{\imath}} \varkappa_\daleth = 1$, *then the below characteristics are obtained.*

Boundedness property: For every $\varkappa$, where

$$P^- \leq PCFHHG_{w,\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) \leq P^+,$$

$$P^+ = \left\{\left(\left[\min \tilde{\dot{z}}_{\sigma(\daleth)}^{-}, \max \tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right], \max \tilde{\vartheta}_{\sigma(\daleth)}\right), \left(\left[\max \tilde{\dot{s}}_{\sigma(\daleth)}^{-}, \min \tilde{\dot{s}}_{\sigma(\daleth)}^{+}\right], \min \tilde{\delta}_{\sigma(\daleth)}\right), \left(\left[\max \tilde{\dot{c}}_{\sigma(\daleth)}^{-}, \min \tilde{\dot{c}}_{\sigma(\daleth)}^{+}\right], \min \tilde{\dot{e}}_{\sigma(\daleth)}\right)\right\},$$

$$P^- = \left\{\left(\left[\max \tilde{\dot{z}}_{\sigma(\daleth)}^{-}, \min \tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right], \min \tilde{\vartheta}_{\sigma(\daleth)}\right), \left(\left[\min \tilde{\dot{s}}_{\sigma(\daleth)}^{-}, \max \tilde{\dot{s}}_{\sigma(\daleth)}^{+}\right], \max \tilde{\delta}_{\sigma(\daleth)}\right), \left(\left[\min \tilde{\dot{c}}_{\sigma(\daleth)}^{-}, \max \tilde{\dot{c}}_{\sigma(\daleth)}^{+}\right], \max \tilde{\dot{e}}_{\sigma(\daleth)}\right)\right\}$$

Idempotency property: If all $P_\daleth = < c_{P_\daleth}, c'_{P'_\daleth}, c''_{P'_\daleth} >, (\daleth = 1, 2, \ldots, \ddot{\imath})$ are equal, *i.e.*, $P_\daleth > P$, then

$$PCFHHA_{w,\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) = P.$$

Monotonicity property: Let

$$P^* = \left\{\left(\left[\tilde{\dot{z}}_{\sigma(\daleth)}^{*-}, \tilde{\dot{z}}_{\sigma(\daleth)}^{*+}\right], \tilde{\vartheta}_{\sigma(\daleth)}^{*}\right), \left(\left[\tilde{\dot{s}}_{\sigma(\daleth)}^{*-}, \tilde{\dot{s}}_{\sigma(\daleth)}^{*+}\right], \tilde{\delta}_{\sigma(\daleth)}^{*}\right), \left(\left[\tilde{\dot{c}}_{\sigma(\daleth)}^{*-}, \tilde{\dot{c}}_{\sigma(\daleth)}^{*+}\right], \tilde{\dot{e}}_{\sigma(\daleth)}^{*}\right)\right\} (\daleth = 1, 2, \ldots, \ddot{\imath})$$

be a group of PCFVs, if

$$\left[\tilde{\dot{z}}_{\sigma(\daleth)}^{-}, \tilde{\dot{z}}_{\sigma(\daleth)}^{+}\right] \leq \left[\tilde{\dot{z}}_{\sigma(\daleth)}^{*-}, \tilde{\dot{z}}_{\sigma(\daleth)}^{*+}\right], \vartheta_{\sigma(\daleth)} \leq \tilde{\vartheta}_{\sigma(\daleth)}^{*}, \left[\tilde{\dot{s}}_{\sigma(\daleth)}^{*-}, \tilde{\dot{s}}_{\sigma(\daleth)}^{*+}\right] \leq \left[\tilde{\dot{s}}_{\sigma(\daleth)}^{-}, \tilde{\dot{s}}_{\sigma(\daleth)}^{+}\right],$$

$$\tilde{\delta}_{\sigma(\daleth)}^{*} \leq \tilde{\delta}_{\sigma(\daleth)}, \left[\tilde{\dot{c}}_{\sigma(\daleth)}^{*-}, \tilde{\dot{c}}_{\sigma(\daleth)}^{*+}\right] \leq \left[\tilde{\dot{c}}_{\sigma(\daleth)}^{-}, \tilde{\dot{c}}_{\sigma(\daleth)}^{+}\right], \tilde{\dot{e}}_{\sigma(\daleth)}^{*} \leq \tilde{\dot{e}}_{\sigma(\daleth)},$$

for all $\urcorner$, then

$$PCFHHA_{w,\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) \leq PCFHHA_{w,\varkappa}(P_1^*, P_2^*, P_3^*, \ldots, P_{\ddot{\imath}}^*).$$

Theorem 4. *The PCFHWA operator is a particular case of the PCFHHA operator.*

Proof. Let $\varkappa = (1/\ddot{\imath}, 1/\ddot{\imath}, \ldots, 1/\ddot{\imath})^T$, and $\tilde{P}_{\sigma_{(\urcorner)}} = P_{\sigma_{(\urcorner)}}$ $(\urcorner = 1, 2, \ldots, \ddot{\imath})$, then

$$\begin{aligned} PCFHHA_{w,\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) &= \varkappa_1 \tilde{P}_{\sigma_{(1)}} \otimes \varkappa_2 \tilde{P}_{\sigma_{(2)}} \otimes \varkappa_3 \tilde{P}_{\sigma_{(3)}} \otimes, \ldots, \otimes \varkappa_{\ddot{\imath}} \tilde{P}_{\sigma_{(\ddot{\imath})}} \\ &= \varkappa_1 P_{\sigma_{(1)}} \otimes \varkappa_2 P_{\sigma_{(2)}} \otimes, \ldots, \otimes \varkappa_{\ddot{\imath}} P_{\sigma_{(\ddot{\imath})}} \\ &= PCFHOWA_{\varkappa}(P_1, P_2, \ldots, P_{\ddot{\imath}}). \end{aligned}$$

Hence proved the result. □

Theorem 5. *The PCFHOWA operator is a particular instance of PCFHHA operator.*

Proof. Let $\varkappa = (1/\ddot{\imath}, 1/\ddot{\imath}, \ldots, 1/\ddot{\imath})^T$, then $\tilde{P}_{\sigma_{(\urcorner)}} = P_{\sigma_{(\urcorner)}}$ $(\urcorner = 1, 2, \ldots, \ddot{\imath})$, thus

$$\begin{aligned} PCFHHA_{w,\varkappa}(P_1, P_2, P_3, \ldots, P_{\ddot{\imath}}) &= \varkappa_1 \tilde{P}_{\sigma_{(1)}} \otimes \varkappa_2 \tilde{P}_{\sigma_{(2)}} \otimes \varkappa_3 \tilde{P}_{\sigma_{(3)}} \otimes, \ldots, \otimes \varkappa_{\ddot{\imath}} \tilde{P}_{\sigma_{(\ddot{\imath})}} \\ &= \varkappa_1 P_{\sigma_{(1)}} \otimes \varkappa_2 P_{\sigma_{(2)}} \otimes, \ldots, \otimes \varkappa_{\ddot{\imath}} P_{\sigma_{(\ddot{\imath})}} \\ &= PCFHOWA_{\varkappa}(P_1, P_2, \ldots, P_{\ddot{\imath}}). \end{aligned}$$

Thus proved. □

MCDM ALGORITHM FOR PICTURE CUBIC FUZZY AOPs

A novel approach is introduced in the context of picture cubic fuzzy averaging AOPs to evaluate MCGDM. In this method, the criteria are assigned real-number weights, and the criterion values are represented as PCFNs. Let $\mathbb{R} = \{\mathbb{R}_1, \mathbb{R}_2, \ldots, \mathbb{R}_{\ddot{\imath}}\}$ denote the set of discrete Altrs, and $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m\}$ represent the criteria to be evaluated alongside their associated WV $\varkappa = (\varkappa_1, \varkappa_2, \ldots \varkappa_{\ddot{\imath}})^T$ with the end goal that $\varkappa_\urcorner \in [0, 1]$ and $\Sigma_{\urcorner=1}^{\ddot{\imath}} \varkappa_\urcorner = 1$. The DMs need to give information about $\mathbb{R}_\top$, which satisfy the criteria also, those which do not satisfy the criteria $\check{C}_\urcorner$ too. The rating of Altrs $\mathbb{R}_\top$ on the basis of criteria $\check{C}_\urcorner$, are given in the form of PCFNs i.e., $\exists : \top_{\top\urcorner} = < c_{\top_\urcorner}, c'_{\top'_\urcorner}, c''_{\top'_\urcorner} > (\urcorner = 1, 2, \ldots, m)$ $(\top = 1, 2, \ldots, \ddot{\imath})$. Let the grade of $\mathbb{R}_\top$ satisfying the criteria $\check{C}_\urcorner$ be indicated by $c_{\top\urcorner}$ $c'_{\top\urcorner}$ represents the neutral membership grade of the alternative $\mathbb{R}$ and $c''_{\top'_\urcorner}$ represents the grade of Altrs $\mathbb{R}_\top$ not satisfying the criteria $\check{C}_\urcorner$, then $c_{\top\urcorner} = \langle [\dot{z}^-_{\top\urcorner}, \dot{z}^+_{\top\urcorner}], \vartheta_{\top\urcorner} \rangle$, $c'_{\top\urcorner} = \langle [\check{s}^-_{\top\urcorner}, \check{s}^+_{\top\urcorner}], \delta_{\top\urcorner} \rangle$ $c''_{\top'_\urcorner} = \langle [\dot{c}^-_{\top\urcorner}, \dot{c}^+_{\top\urcorner}], \dot{e}_{\top\urcorner} \rangle$ with the property that $[\check{s}^-_{\top\urcorner}, \check{s}^+_{\top\urcorner}] \subset [0, 1], [\dot{z}^-_{\top\urcorner}, \dot{z}^+_{\top\urcorner}] \subset [0, 1], \vartheta_{\top\urcorner} : \exists \to [0, 1]$ and $\delta_{\top\urcorner} : \exists \to [0, 1]$. Subject to $\vartheta_{\top\urcorner} + \delta_{\top\urcorner} \leq 1$, $\sup[\dot{z}^-_{\top\urcorner}, \dot{z}^+_{\top\urcorner}] + \sup[\check{s}^-_{\top\urcorner}, \check{s}^+_{\top\urcorner}] \leq 1$. Therefore, a PCF decision matrix D can show MCDM problem, and is presented as follows: $D = (\top_{\top\urcorner})_{\ddot{\imath}\times m} = \left(< c_{\top_\urcorner}, c'_{\top'_\urcorner}, c''_{\top'_\urcorner} >\right)_{\ddot{\imath}\times m}$. Moreover, in this MCGDM way the following steps are stated:

Step 1: Form the decision matrix $D = (\top_{\top\urcorner})_{\ddot{\imath}\times m} = \left(< c_{\top_\urcorner}, c'_{\top'_\urcorner}, c''_{\top'_\urcorner} >\right)_{\ddot{\imath}\times m}$ the criteria can be categorized into two classes: benefit criteria and cost criteria. There is no need of normalization if all the criteria are of the same type. However if D has both cost and benefit criteria, then by the underneath normalization formula, the cost-type criteria can be transformed into benefit-type criteria,

$$S_{\top\urcorner} = \langle v_{\top\urcorner}, t_{\top\urcorner}, \hat{c}_{\top\urcorner} \rangle = \begin{cases} d_{\top\urcorner}, \text{for cost-type criteria} \\ d^c_{\top\urcorner}, \text{for benefit-type criteria} \end{cases},$$

$D^{\ddot{\imath}}_{\top\urcorner}$ is the complement of $D_{\top\urcorner}$, denotes the normalized decision matrix and is presented as follows:

$$D^{\ddot{\imath}} = (s_{\top\urcorner})_{\ddot{\imath}\times m} = (\langle v_{\top\urcorner}, t_{\top\urcorner}, \hat{c}_{\top\urcorner} \rangle)_{\ddot{\imath}\times m}, (\top = 1, 2, \ldots, \ddot{\imath}; \urcorner = 1, 2, \ldots, m).$$

Next, we will utilize the PCFHWA, PCFHOWA, and PCFHHA AOPs in MCDM, having the following steps:
Step 2: By discovering the PCFVs for $\mathbb{R}_\top$ use the proposed AOPs to gain the aggregated values of $\mathbb{R}_\top$, while the criteria WV is $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_{\ddot{\imath}})^T$.
Step 3: We discover the scores of the apparent multitude of values for the positioning of all the Altrs $\mathbb{R}_\top$ by using score function to choose the best $\mathbb{R}_\top$.
Step 4: For choosing the best one, we give ranks to all the Altrs $\mathbb{R}_\top$.

ALGORITHMS FOR DECISION-MAKING

In this section, we introduce two algorithms for decision-making, namely the TOPSIS method and the VIKOR method under the picture cubic fuzzy environment.

Extended TOPSIS method

In practice, the technique of TOPSIS depends on the notion that the best Altr out of the given Altrs will be that which is far from the NIS and close to the PIS. This strategy is founded on the idea of the level of optimality established in an Altr where various criteria represent the concept of the best Altr. The TOPSIS technique has been used in a variety of decision scenarios (Jahanshahloo et al., 2006; Li, 2010). This is as a result of (i) its computational viability, (ii) its importance for dealing with various viable decision-making issues and simplicity, and (iii) its ability to understand. By selecting the lowest value and maximum value, respectively, the NIS $\eth^-$ and the PIS $\eth^+$ can be evaluated based on the aforementioned idea, for each service attribute, across all Altrs. The TOPSIS method can be computed on the below stated steps.

Step 1: Normalize $D = (P_{\top\urcorner})_{\ddot{\imath}\times m} = \left(< c_{\top_\urcorner}, c'_{\top'_\urcorner}, c''_{\top'_\urcorner} >\right)_{\ddot{\imath}\times m}$, The criteria can often be classified into two groups: benefit criteria and cost criteria. If all the criteria are the same type, the normalization step will not be carried out. But if D includes both cost criteria and benefit criteria, into the benefit criteria the rating values of the cost criteria can be transformed using the normalizing approach described below.

$$s_{\top\urcorner} = \langle \mu_{\top\urcorner}, t_{\top\urcorner}, \hat{c}_{\top\urcorner} \rangle = \begin{cases} d_{\top\urcorner}, \text{for cost-type criteria} \\ d^c_{\top\urcorner}, \text{ for benefit-type criteria} \end{cases},$$

$D^{\ddot{\imath}}_{\top\urcorner}$ is the complement of $D_{\top\urcorner}$, denotes the normalized decision matrix and is presented as follows:

$$D^{\ddot{\imath}} = (S_{\top\urcorner})_{\ddot{\imath}\times m} = \left(\langle \mu_{\top\urcorner}, t_{\top\urcorner}, \hat{c}_{\top\urcorner} \rangle\right)_{\ddot{\imath}\times m},$$
$$(\top = 1, 2, \ldots, \ddot{\imath};\ \urcorner = 1, 2, \ldots, m).$$

Step 2: Computing the NIS $\eth^-$ and PIS $\eth^+$, that are stated as,

$$\eth^+ = (\complement_1^+, \complement_2^+, \ldots, \complement_m^+), \eth^- = (\complement_1^-, \complement_2^-, \ldots, \complement_m^-),$$

if there is maximizing type criteria, then

$$\complement_\urcorner^+ = \max\{\complement_{\top\urcorner} / 1 \le \top \le \check{n}\} \text{ and } \complement_\urcorner^- = \min\{\complement_{\ddot{\imath}_\urcorner} / 1 \le \top \le \check{n}\},$$

where if the criteria are of minimizing type, then

$$\complement_\urcorner^+ = \min\{\complement_{\top\urcorner} / 1 \le \top \le \check{n}\} \text{ and } \complement_\urcorner^- = \max\{\complement_{\ddot{\imath}_\urcorner} / 1 \le \top \le \check{n}\},$$

and which are assessed using the score function.
Step 3: For every Altr to $\eth^-$ and $\eth^+$ calculate the distance with criteria WeV $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_m)$.

$$\vartheta_\top^- = \sqrt{\Sigma_{\urcorner=1}^m \varkappa_\urcorner \left(\complement_\urcorner^- - \complement_{\top_\urcorner}\right)^2} \text{ and } \vartheta_\top^+ = \sqrt{\Sigma_{\urcorner=1}^m \varkappa_\urcorner \left(\complement_\urcorner^+ - \complement_{\top_\urcorner}\right)^2}.$$

Step 4: To the ideal solution, by using the structure below evaluate the closeness coefficients by each Altr,

$$cc_\top = \vartheta_\top^- / (\vartheta_\top^- + \vartheta_\top^+)\ (\top = 1, 2, 3, \ldots, \check{n}),$$

obtain the overall closeness coefficients.
Step 5: After ranking the Altrs, we will select the best one using the score of PCFVs.

Extended VIKOR method

The VIKOR technique can simultaneously reduce individual regret and increase group utility, improving the decision's outcome. One of the effective MCDM techniques, on the basis of PCF information, is the VIKOR technique. The proposed strategy is based on the classical VIKOR technique's decision principle. With this approach, the opponent receives the least amount of individual regret and the highest amount of collective benefit of the majority. Additionally, group benefit and personal sorrow can be adjusted by changing the coefficient of the decision mechanism according to actual requirement, which can boost the adaptability of decision-making.

The VIKOR technique entails the actions listed below.
Step 1: If all the criteria are of the same type then there is no need for normalization, otherwise normalize decision matrix.
Step 2: Computing the NIS $\eth^-$ and PIS $\eth^+$, that are stated as follows:

$$\eth^+ = (\complement_1^+, \complement_2^+, \ldots, \complement_m^+), \eth^- = (\complement_1^-, \complement_2^-, \ldots, \complement_m^-),$$

if we are given a maximizing type of criteria, then

$$\complement_\urcorner^+ = \max\{\complement_{\top_\urcorner} / 1 \le \top \le \check{n}\} \text{ and } \complement_\urcorner^- = \min\{\complement_{\top_\urcorner} / 1 \le \top \le \check{n}\},$$

if we are given a minimizing type of criteria, then

$$\complement_\urcorner^+ = \min\{\complement_{\top_\urcorner} / 1 \le \top \le \check{n}\} \text{ and } \complement_\urcorner^- = \max\{\complement_{\top_\urcorner} / 1 \le \top \le \check{n}\},$$

that we obtain by using score function of PCFN.
Step 3: Calculate all $\check{L}_\top$, $\check{O}_\top$, and $R_\top$ values which we can get by using the below equations.

$$\check{L}_\top = \sum_{\top=1}^{m} \frac{\varkappa_\urcorner \vartheta\left(\complement_{\top_\urcorner}, \complement_\top^+\right)}{\vartheta\left(\complement_\top^+, \complement_\top^-\right)}, R_\top = \max_{\top \le \urcorner \le m} \frac{\varkappa_\urcorner \vartheta\left(\complement_{\top_\urcorner}, \complement_\urcorner^+\right)}{\vartheta\left(\complement_\urcorner^+, \complement_\urcorner^-\right)},$$

and

$$\breve{O}_{\top} = \frac{v(\check{L}_{\top} - \check{L}^{*})}{(\check{L}^{-} - \check{L}^{*})} + \frac{(1-v)(R_{\top} - R^{*})}{(R^{-} - R^{*})}.$$

Here $R^{*} = \min R_{\top}$, $R^{-} = \max R_{\top}$, $\check{L}^{*} = \min \check{L}_{\top}$, and $\check{L}^{-} = \max \check{L}_{\top}$.
Step 4: Determine the $\check{L}_{\top}$, $R_{\top}$, and $\breve{O}_{\top}$ values for each Altr and rank them in the decreasing order.
Step 5: Figure out a solution.

Numerical application

A mobility impairment is a condition that hinders one's ability to move, impacting a wide spectrum of activities, from basic motor skills like walking to more intricate tasks involving the manipulation of objects with the hands.

Let us consider an MCDM example related to improving accessibility for the disabled people in a public park. The park management wants to choose the most suitable Altr among the following Altrs to improve the accessibility of the disabled people in the park.

$\mathbb{R}_1$ Accessible restrooms
$\mathbb{R}_2$ Tactile paving
$\mathbb{R}_3$ Wheelchair ramps
$\mathbb{R}_4$ Signage and wayfinding

The park management needs to decide which features to prioritize for enhancement based on multiple criteria. Here are the criteria:

$\hat{H}_1$: Aesthetics: The effect of the feature on the overall aesthetic and natural beauty of the park.
$\hat{H}_2$: Cost: The cost associated with implementing the feature.
$\hat{H}_3$: Safety: Effect of the feature on the safety of disabled people who visit the park.
$\hat{H}_4$: Community engagement: The input and support of local disability community which these features receive.

The park management can use these weights to make an informed decision on which feature to prioritize in their efforts to improve accessibility for people with disabilities in the park. For calculation convenience, we take the criteria WV based on the importance of each criterion $w = (0.4, 0.3, 0.2, 0.1)^T$ and the associated WV of the DMs as $\lambda = (0.5, 0.3, 0.2)^T$ based on the proposed method under PCFHVs as listed in Tables 1–3.

By PCFHWA Opt

Step 1: The DMs' informations are given in Tables 1–3. We will not normalize the criteria as all the criteria are of the same benefit type.

Table 1: First decision maker's data.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	$\begin{pmatrix}([.2,.4],.6),\\([.1,.3],.2),\\([.1,.2],.1)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.1),\\([.1,.4],.3),\\([.1,.2],.5)\end{pmatrix}$	$\begin{pmatrix}([.3,.4],.3),\\([.2,.3],.1),\\([.1,.2],.4)\end{pmatrix}$	$\begin{pmatrix}([.4,.5],.2),\\([.1,.2],.3),\\([.2,.3],.4)\end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix}([.1,.3],.3),\\([.2,.4],.1),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.3,.4],.7),\\([.1,.3],.1),\\([.1,.2],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.2],.3),\\([.3,.4],.2),\\([.2,.3],.4)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.1),\\([.1,.2],.6),\\([.3,.4],.1)\end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix}([.1,.4],.4),\\([.2,.3],.1),\\([.1,.2],.3)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.4),\\([.1,.2],.1),\\([.3,.4],.2)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.5),\\([.1,.4],.1),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.2,.4],.4),\\([.1,.3],.2),\\([.1,.2],.1)\end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix}([.1,.4],.3),\\([.2,.3],.2),\\([.1,.2],.1)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.3),\\([.1,.2],.2),\\([.4,.5],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.3],.4),\\([.2,.3],.1),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.2,.4],.3),\\([.1,.3],.2),\\([.1,.2],.11)\end{pmatrix}$

Table 2: Second decision maker's data.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	$\begin{pmatrix}([.1,.3],.2),\\([.3,.4],.3),\\([.2,.3],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.2],.2),\\([.3,.4],.1),\\([.2,.3],.3)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.2),\\([.1,.2],.1),\\([.3,.4],.4)\end{pmatrix}$	$\begin{pmatrix}([.3,.4],.2),\\([.2,.3],.1),\\([.1,.2],.3)\end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix}([.2,.3],.1),\\([.1,.4],.5),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.1,.5],.3),\\([.2,.3],.1),\\([.1,.2],.5)\end{pmatrix}$	$\begin{pmatrix}([.2,.4],.3),\\([.1,.2],.1),\\([.2,.3],.2)\end{pmatrix}$	$\begin{pmatrix}([.3,.5],.1),\\([.1,.3],.5),\\([.1,.2],.2)\end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix}([.2,.3],.6),\\([.1,.4],.1),\\([.2,.3],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.3],.2),\\([.2,.4],.1),\\([.1,.2],.6)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.7),\\([.1,.4],.1),\\([.1,.2],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.4],.1),\\([.2,.3],.2),\\([.1,.2],.6)\end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix}([.4,.5],.3),\\([.1,.2],.1),\\([.2,.3],.2)\end{pmatrix}$	$\begin{pmatrix}([.1,.3],.1),\\([.2,.3],.2),\\([.1,.3],.4)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.3),\\([.1,.4],.2),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.1,.4],.4),\\([.1,.2],.1),\\([.2,.3],.2)\end{pmatrix}$

Table 3: The third decision maker's information.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	$\begin{pmatrix}([.1,.5],.1),\\([.2,.3],.3),\\([.1,.2],.4)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.2),\\([.1,.4],.5),\\([.2,.3],.3)\end{pmatrix}$	$\begin{pmatrix}([.1,.3],.3),\\([.2,.4],.5),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.1,.2],.2),\\([.2,.3],.3),\\([.1,.3],.4)\end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix}([.1,.3],.2),\\([.2,.4],.5),\\([.1,.2],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.4],.1),\\([.1,.2],.2),\\([.2,.3],.3)\end{pmatrix}$	$\begin{pmatrix}([.2,.3],.3),\\([.1,.4],.1),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.3,.4],.2),\\([.1,.2],.3),\\([.3,.4],.1)\end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix}([.2,.4],.3),\\([.1,.2],.1),\\([.2,.3],.5)\end{pmatrix}$	$\begin{pmatrix}([.3,.5],.2),\\([.1,.2],.5),\\([.2,.3],.1)\end{pmatrix}$	$\begin{pmatrix}([.1,.5],.4),\\([.2,.3],.3),\\([.1,.2],.2)\end{pmatrix}$	$\begin{pmatrix}([.1,.4],.3),\\([.2,.4],.1),\\([.1,.2],.2)\end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix}([.3,.5],.2),\\([.1,.2],.1),\\([.2,.3],.5)\end{pmatrix}$	$\begin{pmatrix}([.2,.4],.1),\\([.1,.2],.4),\\([.2,.3],.3)\end{pmatrix}$	$\begin{pmatrix}([.4,.5],.5),\\([.1,.2],.1),\\([.2,.3],.2)\end{pmatrix}$	$\begin{pmatrix}([.1,.4],.7),\\([.2,.3],.1),\\([.1,.2],.1)\end{pmatrix}$

Let $\beth = 2$, and taking $\varkappa = (.5, .3, .2,)^T$ as WV, using PCFHWA Opt, we have aggregated the data in Tables 1–3, and presented the aggregated data in Table 4.

Table 4: The aggregated data by PCFHWA operator.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	$\begin{pmatrix}([.15,.39],.40),\\([.14,.38],.32),\\([.16,.33],.24)\end{pmatrix}$	$\begin{pmatrix}([.17,.27],.15),\\([.16,.27],.14),\\([.14,.4],.24)\end{pmatrix}$	$\begin{pmatrix}([.23,.35],.27),\\([.21,.35],.27),\\([.16,.28],.14)\end{pmatrix}$	$\begin{pmatrix}([.31,.41],.2),\\([.28,.39],.2),\\([.14,.24],.22)\end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix}([.13,.3],.22),\\([.12,.3],.20),\\([.16,.4],.23)\end{pmatrix}$	$\begin{pmatrix}([.20,.43],.5),\\([.17,.43],.39),\\([.12,.28],.11)\end{pmatrix}$	$\begin{pmatrix}([.15,.28],.3),\\([.14,.27],.3),\\([.17,.33],.14)\end{pmatrix}$	$\begin{pmatrix}([.25,.38],.12),\\([.24,.37],.11),\\([.1,.23],.50)\end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix}([.15,.37],.45),\\([.14,.37],.43),\\([.14,.30],.1)\end{pmatrix}$	$\begin{pmatrix}([.19,.34],.30),\\([.18,.33],.28),\\([.12,.25],.14)\end{pmatrix}$	$\begin{pmatrix}([.18,.34],.55),\\([.17,.28],.28),\\([.11,.38],.12)\end{pmatrix}$	$\begin{pmatrix}([.15,.4],.29),\\([.22,.33],.12),\\([.14,.31],.17)\end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix}([.23,.45],.28),\\([.19,.45],.28),\\([.14,.24],.14)\end{pmatrix}$	$\begin{pmatrix}([.17,.32],.20),\\([.16,.32],.17),\\([.12,.23],.23)\end{pmatrix}$	$\begin{pmatrix}([.19,.34],.39),\\([.16,.33],.38),\\([.14,.30],.12)\end{pmatrix}$	$\begin{pmatrix}([.15,.4],.43),\\([.14,.4],.39),\\([.11,.27],.14)\end{pmatrix}$

Step 2: Taking $\beth = 2$, the aggregated data in Table 4 is again aggregated by using PCFHWA Opt with $\varkappa = (0.4, 0.3, 0.2, 0.1)^T$ as WV; we have the aggregated PCFNs for the Altrs $\mathbb{R}_p$ ($P = 1, \ldots, 4$).

$\mathbb{R}_1 = (([.18, .34], .28), ([.16, .34], .24), ([.15, .33], .21))$
$\mathbb{R}_2 = (([.15, .34], .31), ([.15, .34], .27), ([.14, .33], .20))$
$\mathbb{R}_3 = (([.16, .34], .41), ([.16, .33], .32), ([.12, .30], .12))$
$\mathbb{R}_4 = (([.18, .38], .29), ([.17, .38], .28), ([.13, .25], .16))$

Step 3: Using Definition (2), calculated the scores $S(\mathbb{R}_p)$ of $\mathbb{R}_p$ ($P = 1, \ldots, 4$) as given by;

$$S(\mathbb{R}_{1.}) = 0.28, S(\mathbb{R}_{2.}) = 0.29, S(\mathbb{R}_{3.}) = 0.40, S(\mathbb{R}_{4.}) = 0.38,$$

The PCFNs have been organized in descending order based on their scores, and the best Altr will be selected accordingly, as follows:

$$\mathbb{R}_3 = \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$$

By this ranking, we have found that wheelchair ramps are the best choice for park management. Therefore, the park management should build wheelchair ramps to improve the accessibility for disabled people.

By PCFHOWA Opt

Step 1: The aggregated information of all the DMs by PCFOWA Opt is given in Table 5.

Table 5: Aggregated data by PCFHOWA operator.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	$\begin{pmatrix}([.33,.34],.3),\\([.17,.39],.3),\\([.16,.32],.24)\end{pmatrix}$	$\begin{pmatrix}([.13,.28],.17),\\([.12,.27],.16),\\([.12,.4],.31)\end{pmatrix}$	$\begin{pmatrix}([.28,.33],.28),\\([.16,.32],.27),\\([.17,.32],.23)\end{pmatrix}$	$\begin{pmatrix}([.29,.39],.2),\\([.26,.37],.2),\\([.16,.26],.14)\end{pmatrix}$
$\mathbb{R}_2$	$\begin{pmatrix}([.23,.3],.19),\\([.12,.3],.17),\\([.16,.4],.37)\end{pmatrix}$	$\begin{pmatrix}([.20,.33],.29),\\([.17,.37],.24),\\([.12,.27],.11)\end{pmatrix}$	$\begin{pmatrix}([.18,.30],.3),\\([.17,.32],.3),\\([.12,.28],.11)\end{pmatrix}$	$\begin{pmatrix}([.27,.42],.12),\\([.26,.41],.11),\\([.1,.14],.48)\end{pmatrix}$
$\mathbb{R}_3$	$\begin{pmatrix}([.18,.37],.41),\\([.17,.36],.39),\\([.1,.26],.1)\end{pmatrix}$	$\begin{pmatrix}([.23,.40],.26),\\([.21,.34],.24),\\([.11,.23],.23)\end{pmatrix}$	$\begin{pmatrix}([.17,.36],.38),\\([.16,.35],.45),\\([.12,.36],.14)\end{pmatrix}$	$\begin{pmatrix}([.15,.4],.31),\\([.14,.2],.28),\\([.14,.32],.16)\end{pmatrix}$
$\mathbb{R}_4$	$\begin{pmatrix}([.23,.45],.28),\\([.19,.44],.27),\\([.14,.24],.14)\end{pmatrix}$	$\begin{pmatrix}([.17,.35],.14),\\([.16,.38],.12),\\([.12,.22],.28)\end{pmatrix}$	$\begin{pmatrix}([.28,.30],.42),\\([.25,.39],.31),\\([.11,.26],.2)\end{pmatrix}$	$\begin{pmatrix}([.13,.4],.34),\\([.12,.3],.49),\\([.14,.27],.1)\end{pmatrix}$

Step 2: Taking $\beth = 2$, using the data in Table 5, utilizing PCFHOWA Opt, with WV $\varkappa = (.3, .25, .2, .15, .1)^T$, we have the collective PCFNs for the Altrs $\mathbb{R}_p$ ($P = 1, \ldots, 4$), which are presented as follows:

$\mathbb{R}_1 = (([0.28, 0.35], 0.27), ([0.19, 0.35], 0.24), ([0.15, 0.30], 0.20))$
$\mathbb{R}_2 = (([0.21, 0.37], 0.33), ([0.18, 0.35], 0.22), ([0.12, 0.26], 0.21))$
$\mathbb{R}_3 = (([0.18, 0.34], 0.44), ([0.17, 0.35], 0.42), ([0.11, 0.30], 0.14))$
$\mathbb{R}_4 = (([0.15, 0.30], 0.40), ([0.19, 0.40], 0.38), ([0.12, 0.25], 0.13))$

Step 3: Calculated the scores $S(\mathbb{R}_i)$ of $\mathbb{R}_i$ ($i = 1, 2, 3, 4$) by using Definition (2) as given by

$$S(\mathbb{R}_{1.}) = 0.34, S(\mathbb{R}_{2.}) = 0.35, S(\mathbb{R}_{3.}) = 0.46, S(\mathbb{R}_{4.}) = 0.44,$$

The PCFNs have been organized in descending order based on their scores, and the best Altr will be selected accordingly, as follows:

$$\mathbb{R}_{3.} > \mathbb{R}_{4.} > \mathbb{R}_{2.} > \mathbb{R}_{1.}$$

By this ranking, we have found that wheelchair ramps are the best choice for park management. Therefore, the park management should build wheelchair ramps to improve the accessibility for disabled people.

By PCFHHA Opt

Step 1: The DMs have given their decisions in Tables 1–3. Apply the formula,

$$\tilde{P}_\daleth = \ddot{\iota} w_\daleth P_\daleth = \langle([\tilde{z}^-_\daleth, \tilde{z}^+_\daleth], \tilde{\vartheta}_\daleth), ([\tilde{s}^-_\daleth, \tilde{s}^+_\daleth], \tilde{\delta}_\daleth), ([\tilde{c}^-_\daleth, \tilde{c}^+_\daleth], \tilde{e}_\daleth)\rangle, (\daleth = 1, 2, \ldots, \ddot{\iota})$$

to the information given in Tables 1–3, taking $\varkappa = (0.5, 0.3, 0.2)^T$ as WV to be multiplied to Tables 1–3, respectively. Then aggregate that data, and the aggregated information by using PCFHHA Opt is given in Table 6, where $\varkappa = (0.4, 0.4, 0.2)^T$ is the WV based on the varying significance or influence of all the DMs.

Table 6: Aggregated information by PCFHHA operator.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	([.07, .39], .45), ([.19, .32], .25), ([.14, .23], .16)	([.08, .28], .14), ([.17, .39], .27), ([.16, .25], .41)	([.24, .35], .27), ([.19, .31], .23), ([.16, .25], .35)	([.33, .43], .20), ([.19, .29], .21), ([.14, .26], .36)
$\mathbb{R}_2$	([.20, .31], .18), ([.16, .39], .37), ([.11, .22], .32)	([.22, .45], .55), ([.14, .28], .13), ([.13, .22], .26)	([.17, .34], .31), ([.17, .31], .14), ([.19, .29], .28)	([.21, .44], .11), ([.11, .27], .48), ([.21, .31], .17)
$\mathbb{R}_3$	([.17, .36], .53), ([.14, .33], .11), ([.19, .29], .24)	([.18, .33], .25), ([.17, .29], .34), ([.26, .36], .26)	([.19, .34], .63), ([.14, .38], .16), ([.11, .22], .16)	([.11, .39], .31), ([.17, .33], .18), ([.11, .21], .24)
$\mathbb{R}_4$	([.23, .46], .30), ([.15, .25], .15), ([.16, .25], .22)	([.14, .32], .17), ([.17, .27], .25), ([.22, .38], .30)	([.19, .34], .37), ([.14, .33], .17), ([.14, .24], .22)	([.15, .39], .40), ([.15, .29], .16), ([.14, .23], .14)

Step 2: Applying the formula $\tilde{P}_\daleth = \ddot{\iota} w_\daleth P_\daleth = \langle([\tilde{z}^-_\daleth, \tilde{z}^+_\daleth], \tilde{\vartheta}_\daleth), ([\tilde{s}^-_\daleth, \tilde{s}^+_\daleth], \tilde{\delta}_\daleth), ([\tilde{c}^-_\daleth, \tilde{c}^+_\daleth], \tilde{e}_\daleth)\rangle, (\daleth = 1, 2, \ldots, 5)$ to the data present in Table 6, taking $w = (0.2, 0.2, 0.3, 0.3)^T$ as WV of $\mathbb{R}_p$, the result of which is given in Table 7.

Table 7: Again weight multiplied to Table 6.

	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$
$\mathbb{R}_1$	([.01, .08], .09), ([.4, .3], .2), ([.1, .3], .4)	([.01, .05], .02), ([.2, .4], .3), ([.3, .2], .4)	([.07, .1], .08), ([.4, .1], .3), ([.2, .3], .1)	([.1, .1], .06), ([.5, .2], .3), ([.1, .4], .2)
$\mathbb{R}_2$	([.04, .06], .03), ([.5, .1], .2), ([.1, .4], .3)	([.04, .09], .1), ([.3, .2], .1), ([.1, .4], .3)	([.05, .1], .09), ([.4, .3], .1), ([.3, .2], .1)	([.06, .1], .03), ([.4, .3], .2), ([.2, .1], .4)
$\mathbb{R}_3$	([.03, .07], .1), ([.4, .2], .1), ([.3, .1], .5)	([.03, .06], .05), ([.4, .3], .2), ([.1, .5], .2)	([.05, .1], .2), ([.4, .1], .2), ([.3, .2], .4)	([.03, .1], .09), ([.4, .3], .1), ([.1, .2], .3)
$\mathbb{R}_4$	([.04, .09], .06), ([.5, .3], .1), ([.1, .4], .3)	([.02, .06], .03), ([.4, .1], .2), ([.1, .3], .4)	([.05, .1], .1), ([.3, .1], .2), ([.4, .3], .2)	([.04, .1], .1), ([.3, .2], .4), ([.4, .1], .2)

Again utilizing the PCFHHA Opt, we get the collective PCFNs for the Altrs $\mathbb{R}_p$ (i =1, 2, 3, 4) as given below, where $\varkappa = (0.4, 0.3, 0.2, 0.1)^T$ is the criteria WV,

$\mathbb{R}_1$ = (([.03, .08], .06), ([.3, .2], .2), ([.3, .3], .2))
$\mathbb{R}_2$ = (([.04, .08], .06), ([.4, .2], .2), ([.3, .2], .4))
$\mathbb{R}_3$ = (([.4, .07], .10), ([.3, .2], .4), ([.2, .30], .2))
$\mathbb{R}_4$ = (([.03, .08], .06), ([.3, .3], .2), ([.2, .2], .3))

Step 3: Using Definition (2), calculate the scores $S(P_p)$ of $P_p(P = 1, \ldots, 4)$ as follows:

$$S(\mathbb{R}_1) = 0.023,\ S(\mathbb{R}_2) = 0.026,\ S(\mathbb{R}_3) = 0.260,\ S(\mathbb{R}_4) = 0.090,$$

Step 4: The PCFNs have been organized in descending order based on their scores, and the best Altr will be selected accordingly, as follows:

$$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$$

It becomes evident from the comparative analysis and the ranking of Altrs based on their score values that $\mathbb{R}_3$ exhibits a notably higher score. This outcome is further illustrated in Table 8 and Figure 1.

By this ranking, we have found that wheelchair ramps are the best choice for park management. Therefore, the park management should build wheelchair ramps to improve the accessibility for disabled people.

Table 8: Comparative study and ranking of the alternatives.

Operators	$S(\mathbb{R}_1)$	$S(\mathbb{R}_2)$	$S(\mathbb{R}_3)$	$S(\mathbb{R}_4)$	Ranking
PCFHWA	0.28	0.29	0.40	0.38	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
PCFHOWA	0.34	0.35	0.46	0.44	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
PCFHHA	0.02	0.02	0.26	0.09	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$

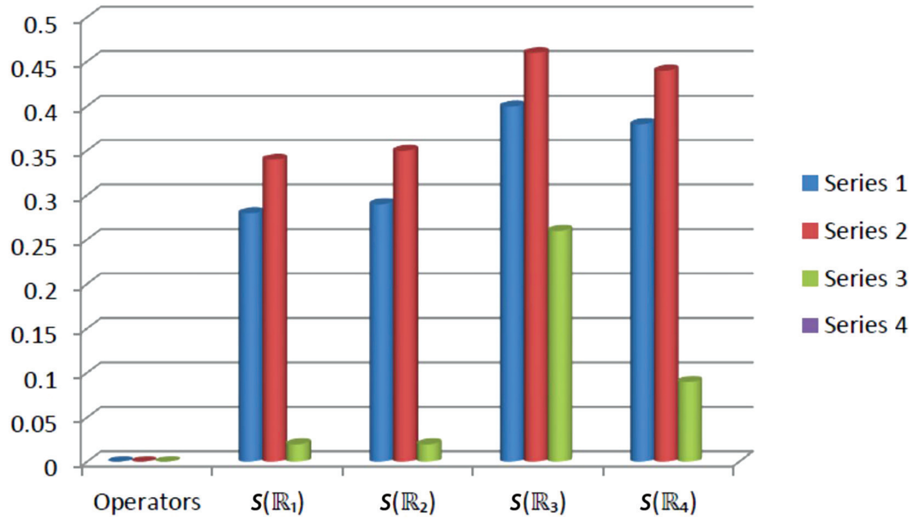

Figure 1: Ranking of the alternatives.

By the TOPSIS method

Step 1: In Table 4, data will not be normalized because all the criteria are of the same type, i.e. benefit type.
Step 2: Calculate the NIS ð⁻ and PIS ð⁺, by utilizing the formula stated below

$$\eth^- = (\mathsf{C}_1^-, \mathsf{C}_2^-, \ldots, \mathsf{C}_m^-), \eth^+ = (\mathsf{C}_1^+, \mathsf{C}_2^+, \ldots, \mathsf{C}_m^+),$$

where

$$\mathsf{C}_{\urcorner}^+ = \max\left\{\mathsf{C}_{\top\urcorner} / 1 \le \top \le 4\right\}, \mathsf{C}_{\urcorner}^- = \min\left\{\mathsf{C}_{\top\urcorner} / 1 \le \top \le 4\right\},$$

which are evaluated by using score function of PCFNs.
Step 3: Evaluate the distance for each Altr, to $\top^+$ and $\top^-$ by utilizing the proposed distance measures with criteria WeV $\varkappa = (\varkappa_1, \varkappa_2, \ldots, \varkappa_m) = (0.4, 0.3, 0.2, 0.1)^T$, *i.e.*

$$\vartheta_{\top}^- = \sqrt{\Sigma_{\urcorner=1}^m \varkappa_{\urcorner}\left(\mathsf{C}_{\urcorner}^- - \mathsf{C}_{\top\urcorner}\right)^2} \text{ and } \vartheta_{\top}^+ = \sqrt{\Sigma_{\urcorner=1}^m \varkappa_{\urcorner}\left(\mathsf{C}_{\urcorner}^+ - \mathsf{C}_{\top\urcorner}\right)^2}.$$

Step 4: By using the structure proposed below, by each Altr to the ideal solution calculate the closeness coefficients

$$cc_{\top} = \vartheta_{\top}^- / (\vartheta_{\top}^- + \vartheta_{\top}^+)(\top = 1, 2, 3, \ldots, 4),$$

to get overall closeness coefficients.
Step 5: Utilize the PCFNs' score function to rank the Altrs. Ranking is provided as follows:

$$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_1 > \mathbb{R}_4$$

Table 9: Ranking of the Altrs.

Altrs	Distance for Altr to $\vartheta^{+}_{\top}$	Distance for Altr to $\vartheta^{-}_{\top}$	The closeness coefficients of Altrs ($cc_{\top}$) to the ideal solution	Ranking
$\mathbb{R}_1$	0.016	0.018	0.534	4
$\mathbb{R}_2$	0.009	0.012	0.567	3
$\mathbb{R}_3$	0.013	0.029	0.683	1
$\mathbb{R}_4$	0.0123	0.2096	0.641	2

Abbreviation: Altr, alternative.

In Table 9, ranking of all Altrs is given. $\mathbb{R}_3$ having the largest closeness coefficient is the best one.

By the VIKOR method

By using the VIKOR method we address the numerical issues. $\varkappa = (0.4, 0.3, 0.2, 0.1)^T$ as the criteria WeV the VIKOR method has the following steps:
Step 1: For data presented in Tables 1–3, all the criteria is of the benefit type so no need to normalize the data.
Step 2: By the below stated formula, evaluate the $\eth^{+}$ and $\eth^{-}$.

$$\eth^{+} = (\mathfrak{C}_1^{+}, \mathfrak{C}_2^{+}, \mathfrak{C}_3^{+}, \ldots, \mathfrak{C}_4^{+}), \eth^{-} = (\mathfrak{C}_1^{-}, \mathfrak{C}_2^{-}, \mathfrak{C}_3^{-}, \ldots, \mathfrak{C}_4^{-})$$

where $\mathfrak{C}_{\urcorner}^{+} = \max\left\{\mathfrak{C}_{\top\urcorner} / 1 \le \top \le 4\right\}$ and $\mathfrak{C}_{\urcorner}^{-} = \min\left\{\mathfrak{C}_{\top\urcorner} / 1 \le \top \le 4\right\}$, which are assessed by score function of PCFNs.

Step 3: By utilizing the below stated formulae, compute all $\breve{O}_{\top}$, $R_{\top}$, and $\check{L}_{\top}$ values

$$\check{L}_{\top} = \sum_{\urcorner=1}^{m} \frac{\varkappa_{\urcorner}\vartheta\left(\mathfrak{C}_{\top\urcorner}, \mathfrak{C}_{\urcorner}^{+}\right)}{\vartheta\left(\mathfrak{C}_{\urcorner}^{+}, \mathfrak{C}_{\urcorner}^{-}\right)}, R_{\top} = \max_{\top \le \urcorner \le m} \frac{\varkappa_{\urcorner}\vartheta\left(\mathfrak{C}_{\top\urcorner}, \mathfrak{C}_{\urcorner}^{+}\right)}{\vartheta\left(\mathfrak{C}_{\urcorner}^{+}, \mathfrak{C}_{\urcorner}^{-}\right)},$$

and

$$\breve{O}_{\top} = \frac{v(\check{L}_{\top} - \check{L}^{*})}{(\check{L}^{-} - \check{L}^{*})} + \frac{(1-v)(R_{\top} - R^{*})}{(R^{-} - R^{*})}.$$

Suppose $v = 0.5$, then Table 10 presents the results. Also

$$\check{L}^{*} = 0.633, \check{L}^{-} = 1.054, R^{*} = 0.260, R^{-} = 0.534.$$

Step 4: By assembling all the values of $\check{L}_{\top}$, $R_{\top}$, and $\breve{O}_{\top}$, in a decreasing order, rank the Altrs. The ranking of the values of $\breve{O}_{\top}$ is as follows:

$$\breve{O}_4 > \breve{O}_1 > \breve{O}_2 > \breve{O}_3.$$

Step 5: It is clear from the ranking result that $\breve{O}_3$ is the best choice. By measure the minimum value $\breve{O}_3$, is the compromise solution.

Table 10: Ranking of the Altrs.

Altrs	$\check{L}_{\top}$	$R_{\top}$	$\breve{O}_{\top}$	Rank
$\mathbb{R}_1$	0.749	0.490	0.919	4
$\mathbb{R}_2$	0.824	0.371	0.772	3
$\mathbb{R}_3$	0.357	0.230	0	1
$\mathbb{R}_4$	0.823	0.3	0.633	2

Abbreviation: Altr, alternative.

In Table 10, all the Altr ranking is $\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$. Thus, $\mathbb{R}_3$ is the best one.

SENSITIVITY ANALYSIS

In the VIKOR method, the basic to the ranking results is v, the decision-making coefficient. Consequently, in these MCGDM algorithms the sensitivity analysis is carried out to assess the stability of our suggested method. For each v at 0.1 intervals from 0 to 1, we compute the comparing compromise solution in order to assess the effect of various v on the ranking result. Table 11 demonstrates the sensitivity analysis for choosing a greenhouse location. For each and every tested value of v we get one ranking result, given below:

$$\mathbb{R}_3 > \mathbb{R}_2 > \mathbb{R}_1 > \mathbb{R}_4$$

It is clear that $\mathbb{R}_3$ is the optimal solution. The sensitivity analysis is presented in Table 11.

Table 11: Sensitivity analysis.

V	$\check{O}_1$	$\check{O}_2$	$\check{O}_3$	$\check{O}_4$	Ranking
0.1	0.984	0.589	0	0.342	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.2	0.968	0.634	0	0.415	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.3	0.952	0.680	0	0.488	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.4	0.936	0.726	0	0.561	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.5	0.919	0.772	0	0.633	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.6	0.903	0.817	0	0.706	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.7	0.887	0.863	0	0.779	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
0.8	0.871	0.909	0	0.852	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_1 > \mathbb{R}_2$
0.9	0.855	0.954	0	0.925	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$
1	0.839	1	0	0.908	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$

Thus, to improve the accessibility for disabled people all the methods have been successfully applied.

COMPARISON ANALYSIS

In this section, our recommended advance fuzzy AOPs are compared with previous AOPs. We resolved our developed problem in the Numerical Application section by applying the proposed strategy presented by Khoshaim et al. (2021). Utilizing the criteria WV $\varkappa = (.4, .3, .2, .1)^T$ and all the steps of the Khoshaim et al.'s (2021) technique, we arrived at the following ranking. In Table 12, the comparison of our proposed three strategies with the existing strategies in Khoshaim et al. (2021) is stated. The ranking of our proposed methods and the technique presented in Khoshaim et al. (2021) are similar.

Table 12: Comparison analysis.

Operators	$S(\mathbb{R}_1)$	$S(\mathbb{R}_2)$	$S(\mathbb{R}_3)$	$S(\mathbb{R})$	Ranking
PCFHWA	0.28	0.29	0.040	0.38	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
PCFHOWA	0.34	0.35	0.46	0.44	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
PCFHHA	0.02	0.02	0.26	0.09	$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
TOPSIS method					$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
VIKOR method					$\mathbb{R}_3 > \mathbb{R}_4 > \mathbb{R}_2 > \mathbb{R}_1$
PCFWA (Khoshaim et al., 2021)	0.085	0.074	0.085	0.084	$\mathbb{R}_3 = \mathbb{R}_1 > \mathbb{R}_4 > \mathbb{R}_2$
PCFOWA (Khoshaim et al., 2021)	0.101	0.081	0.105	0.055	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_2 > \mathbb{R}_4$
PCFHA (Khoshaim et al., 2021)	0.083	0.062	0.083	0.061	$\mathbb{R}_3 = \mathbb{R}_1 > \mathbb{R}_2 > \mathbb{R}_4$
PCFWG (Khoshaim et al., 2021)	0.105	0.081	0.105	0.055	$\mathbb{R}_3 = \mathbb{R}_1 > \mathbb{R}_2 > \mathbb{R}_4$
PCFOWG (Khoshaim et al., 2021)	0.081	0.062	0.083	0.061	$\mathbb{R}_3 > \mathbb{R}_1 > \mathbb{R}_2 > \mathbb{R}_4$
PCFHG (Khoshaim et al., 2021)	0.086	0.083	0.085	0.083	$\mathbb{R}_1 > \mathbb{R}_3 > \mathbb{R}_4 = \mathbb{R}_2$

Abbreviations: PCFHG, picture cubic fuzzy hybrid geometric; PCFOWA, picture cubic fuzzy Hamacher order weighted averaging; PCFOWG, picture cubic fuzzy order weighted geometric; PCFWA, picture cubic fuzzy Hamacher weighted averaging; PCFWG, picture cubic fuzzy weighted geometric; TOPSIS, technique for order preference by similarity to ideal solution; VIKOR, VIseKriterijumska Optimizacija I Kompromisno Resenje.

This observation underscores the limited capabilities of the existing AOPs. In contrast, the employment of PCF Hamacher averaging AOPs yields more precise outcomes, as they do not possess such restrictions. The consistency of the proposed approaches is checked by leading a comparative examination with the existing AOPs. The PCFS, in the context of decision-making, plays a crucial role in managing vagueness and uncertainty by expressing the cubic and picture fuzzy information simultaneously. The PCF models are proficient in practices and more accommodating in taking care of true issues when contrasted with other existing fuzzy models.

CONCLUSION

Generally, engaging with nature and nature-related activities can promote the health of individuals with mobility impairments. These benefits encompass physical health, mental well-being, and social interaction. However, obstacles frequently hinder the access to nature for people facing mobility impairments. In this article, we have presented an MCDM example related to improving accessibility for disabled people in a public park. Since AOPs play a crucial role in decision-making, we established aggregation techniques for PCFNs and established a series of AOPs, such as PCFHOWA Opt, PCFWA Opt, and PCFHHA Opt. We discussed some essential properties like boundary, monotonicity, and idempotency and researched the connections among these developed AOPs. We developed a MAGDM model dependent on the proposed AOPs. The Hamacher AOPs are extended to PCFNs, and an exhaustive conversation is introduced to dissect the significant outcomes and predominant properties of the proposed AOPs. The TOPSIS and VIKOR techniques are extended for PCFNs. We have conducted a comparison analysis between the existing AOPs and our proposed AOPs, to demonstrate the credibility, utility, and efficiency of our innovative approaches. Our novel approach in group decision-making stands out from previous methods because it incorporates PCF information, thereby preventing any information gaps within the process. Consequently, it proves to be an effective and viable solution for real-world decision-making applications.

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All authors actively contributed to every phase of the research, and they have reviewed and endorsed the final manuscript.

CONFLICTS OF INTEREST

The authors have no conflict of interest.

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The authors of this article did not conduct any research involving human participants or animals.

DATA AVAILABILITY

To support this study, no data were used related to humans or animals.

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