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# A theorem on separated transformations of basis vectors of polynomial space and its applications in special polynomials and related $\mathfrak{sl}(2,R)$ Lie algebra

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**abstract** The present paper introduces a method of basis transformation of a vector space that is specifically applicable to polynomials space and differential equations with certain polynomials solutions such as Hermite, Laguerre and Legendre polynomials. The method based on separated transformations of vector space basis by a set of operators that are equivalent to the formal basis transformation and connected to it by linear combination with projection operators. Applying the Forbenius covariants yields a general method that incorporates the Rodrigues formula as a special case in polynomial space. Using the Lie algebra modules, specifically $\mathfrak{sl}(2,R)$, on polynomial space results in isomorphic algebras whose Cartan sub-algebras are Hermite, Laguerre and Legendre differential operators. Commutation relations of these algebras and Baker-Campbell-Hausdorff formula gives new formulas for special polynomials.

**Keywords** special polynomials, Hermite, Laguerre, Legendre, differential operators, Lie algebra, Baker-Campbell-Hausdorff formula, separated basis transformation, Forbenius covariant, Rodrigues formula, differential equations

## 1. Introduction

In mathematical physics and specifically quantum mechanics, the solution of many problems requires solving the differential equations and their eigenvalues problem . Hermite, Laguerre and Legendre polynomials and related differential equations are among the most applicable eigenvalues problem in physics and mathematics [1,2]. Schrodinger equation for hydrogen atom reduces to Legendre differential equation and quantum harmonic oscillator requires Hermite polynomials and related differential equation. The well-known Rodrigues formula yields the solutions (eigenfunctions) of many of these differential equations [3]. In this paper we interpret the Rodrigues formula as the transformation of some specific basis in polynomial space to another polynomials (eigenfunctions) of associated differential equations. This approach is feasible, provided that the transformation operator to be considered as a set of operators acting on each basis separately. It is shown that the overall action of these operators is equivalent to a single linear operator. As an example, the change of basis vectors in two dimension can be made by a matrix of rank 2. The action of this matrix could be equivalent with the actions of two different matrices that acts on each basis separately. The relation between these operators achieved by applying projection operators as is proved in the section 2. The connection between separated basis transformation and umbral composition has been revealed by a theorem in section 2. It is proved that by knowing the first two polynomials of Hermite , Laguerre and Legendre polynomials the related differential equations could be retrieved by using the method based on the separated transformation of original basis and Forbenius covariants as projection operators.

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The examples in section 2 clarifies the details of this method. By using the Rodrigues formula as the separated operators acting on the original basis, we acquire the form of related differential equations. In section 3, we introduce the Lie algebra modules on vector space of polynomials. The $\mathfrak{sl}(2,R)$ and $\mathfrak{sl}(2,c)$ has been known as the Lie algebras connected to symmetries in polynomial and monomial space [4,5]. We prove that the conjugation (similarity transformation) of generators of these Lie algebras, yields isomorphic algebras that their Cartan subalgebras are Hermite, Laguerre and Legendre differential operators. The raising and lowering operators has been introduced in section sections 3.1 and 3.2. In section 3.8, applying Baker-Campbell-Hausdorff formula [6] on the basis of these isomorphic algebras, gives new relations of Hermite and Laguerre polynomials and their generating functions. Section 3.6 proves and represents a general form of differential-operator representations of $\mathfrak{sl}(2,R)$. In section 3.7, we propose a technique for solutions of differential equations based on raising operators acquired from associated Lie algebras.

## 2. Separated operators of basis transformation

Let $\mathbb{V}$ be a n-dimensional vector space with basis vectors $e_1, e_2, \dots e_n$ . The linear operator that transforms these basis to another basis $e'_1, e'_2, \dots e'_n$ , normally is defined as a unique linear operator $O$ in the matrix form. In present theory we define a set of linear operators $O_1, O_2, \dots O_n$ ; each one acts **separately** on the corresponding basis as follows:

$$O_i e_i = e'_i \tag{1}$$

The result of the action of operator $O$ and the set of $O_i$ on the initial basis are the same, but $O_i$ as separated basis transformations allow to choose a wide range of $O_i$ operators whose overall transformations are equivalent to the operation of $O$. On the other hand, in many problems such as Rodrigues type formulas the separated basis transformation $O_i$ are more accessible than overall operator $O$. In the context of differential operators, $O_i$ could be regarded as the operators that transform the initial basis (monomials) in polynomial space to another basis as for example we observe in Rodrigues formula for Laguerre polynomials as the solutions (eigenfunctions) for Laguerre differential equation:

$$\mathbb{L}_n = \frac{1}{n!}(D-1)^n x^n$$

This equation can be interpreted as the transformation of initial basis $(1, x, x^2, x^3, \dots)$ to new basis $\mathbb{L}_n$ (Laguerre polynomials) by the action of the operator

$$O_n = \frac{1}{n!}(D-1)^n$$

Respect to Rodrigues formula, for all related differential equations such as Legendre, Chebyshev and Bessel Equations, there are separate and independent operators for each basis. Therefor we can apply a set of operators $O_i$ instead of a unique operator $O$ to transform the initial basis in polynomials space. This method obviates the need to find the unique linear operator $O$ with the same action on all initial bases. We will show the relation between $O_i$ and $O$ in proposition 2.1 after defining the projection operators as follows.

Let introduce the projection operators $P_1, P_2, \dots, P_n$ by the definition:

$$P_i V = V_i e_i \quad ; \quad P_i e_j = \delta_{ij}\, e_i \tag{2}$$

Where $V \in \mathbb{V}$ is a vector expanded as:

$$V = \sum_i V_i\, e_i \tag{3}$$

From (2) we have: $\sum_i P_i = I$ ; $P_i P_j = P_i \delta_{ij}$
as the main condition for projection operators ($I$ is identity operator).

**Remark 1**
projection operators defined in (2), are linear operators.
We show that by basis transformations according to (1), the projection operators $P_i$ are transformed as:

$$P_i' = O_i P_i (\sum_j O_j P_j)^{-1} \quad (4)$$

**Theorem 2.1**: the generalized form of projection operator under **separated** basis transformations $O_i e_i = e'_i$ is:

$$P_i' = O_i P_i (\sum_j O_j P_j)^{-1}$$

**Proof**: respect to basis transformation $O_i e_i = e'_i$ and (2) we have:

$$e'_i = P_i' e'_i = P_i' O_i e_i = P_i' (\sum_j O_j P_j) e_i \quad (5)$$

Again, with substitution $e'_i = O_i e_i$ (5) reads as:

$$O_i e_i = P'_i (\sum_j O_j P_j) e_i \quad (6)$$

It is valid for all $e_i$. with identity $O_i e_i = O_i P_i e_i$ , equation (6) becomes:

$$O_i P_i = P'_i (\sum_j O_j P_j) \quad (7)$$

Or: $P_i' = O_i P_i (\sum_j O_j P_j)^{-1}$
As we expected.

**Proposition 2.1** The transformation of projection operator defined in (4) is equivalent to a similarity transformation

$$P_i' = (\sum_k O_k P_k) P_i (\sum_j O_j P_j)^{-1} = O_i P_i (\sum_j O_j P_j)^{-1} \quad (8)$$

**Proof** : expansion of the first two terms on left side considering $P_i P_j = P_i \delta_{ij}$ yields:

$$(\sum_k O_k P_k) P_i = O_i P_i$$

Therefor we have the right side of (8).
Equation (8) implies the similarity transformation of $P_i$ under the basis transformation made by operator $O = \sum_i O_i P_i$ . Therefor the operator $O$ is the linear operator for transforming all basis $e_i$. Operator $O$ also yields transformations of all linear operators $\mathcal{O}$ in vector space $\mathbb{V}$ under basis transformation $e'_i = O e_i$ by the similarity transformation

$$\mathcal{O}' = O \mathcal{O} O^{-1} = (\sum_k O_k P_k) \mathcal{O} (\sum_j O_j P_j)^{-1}$$

**Remark 2**
Equation (4) meets the projection operator conditions:

a) $\sum_i P_i' = \sum_i O_i P_i (\sum_j O_j P_j)^{-1} = (\sum_i O_i P_i)(\sum_j O_j P_j)^{-1} = I$ (9)

$$P_i' = O P_i O^{-1} = \sum_k O_k P_k) P_i (\sum_j O_j P_j)^{-1} = O_i P_i (\sum_j O_j P_j)^{-1}$$

The action of $O$ and $O_j$ on a basis $e_j$ is the same.

$$O e_j = (\sum_i O_i P_i) e_j = O_j P_j e_j = O_j e_j = e'_i$$

This implies that the action of $O$ and $O_j$ on $e_j$ is equivalent.

b) Respect to (4) and (9) we conclude:

$$P_i' P_j' = O_i P_i (\sum_k O_k P_k)^{-1} O_j P_j (\sum_l O_l P_l)^{-1} = [(\sum_k O_k P_k) P_i (\sum_j O_j P_j)^{-1}][(\sum_k O_k P_k) P_j (\sum_j O_j P_j)^{-1}] \quad (10)$$

Middle terms in right side of (7) reduce to identity operator and thus:
$P_i' P_j' = (\sum_k O_k P_k) P_i P_j (\sum_j O_j P_j)^{-1}$ (11)
Recalling $P_i P_j = P_i \delta_{ij}$ and (4) we obtain:

$$P_i' P_j' = O_i P_i (\sum_j O_j P_j)^{-1} \delta_{ij} = P_i' \delta_{ij}$$

This proves the *idempotency* of $P_i'$ i.e., $P_i' P_i' = P_i'$
Where $P_i'$ denoted as posterior probability analogy.
Equation (4) is the unique formula for $P_i'$ and other forms in spite of their validity for satisfaction of projection operator conditions i.e., equation (3), are not the right candidates. As an example, we may propose this formula for $P_i'$:

$$P_i' = (\sum_j P_j O_j)^{-1} P_i O_i \quad (12)$$

It is straightforward to investigate that this definition is compatible with conditions (3) but if we multiply both sides by $e'_i$ we obtain:

$$(\sum_j P_j O_j) P_i' e'_i = P_i O_i e'_i$$

Respect to (1) and (2) we get:

$$(\sum_j P_j O_j) e'_i = P_i O_i e'_i$$

One of the solutions results in a false outcome:

$$(\sum_j P_j O_j) = P_i O_i$$

**Proposition 2.2** The product of projection operators is associative.

**Proof** : Let projection operators $P_i'$ and $P_i''$ correspond to $O_i$ and $O_i'$ as transformation groups of coordinates i.e.

$$P_i' = O_i P_i (\sum_j O_j P_j)^{-1}$$

And $$P_i'' = O_i' P_i' (\sum_j O_j' P_i')^{-1}$$

Substitution of first relation into above equation results in:

$$P_i'' = O_i' O_i P_i (\sum_j O_j P_j)^{-1} \; [(\sum_j O_j' O_j P_j)(\sum_j O_j P_j)^{-1})]^{-1}$$

$$P_i'' = O_i' O_i P_i (\sum_j O_j P_j)^{-1} [(\sum_j O_j P_j)(\sum_j O_j' O_j P_j)^{-1}]$$

After vanishing of two central terms:

$$P_i'' = O_i' O_i P_i (\sum_j O_j' O_j P_j)^{-1}$$

This is compatible with (4) by replacing $O_i$ with $O_i'' = O_i' O_i$. Therefore, the corresponding projection operator for two consecutive transformation $O_i$ and $O_i'$ is equivalent the projection operator of $O_i'' = O_i' O_i$ transformation.

$$P_i'' = O_i'' P_i (\sum_j O_i'' P_j)^{-1}$$

**Remark 3**

As is proved, the operators $\sum_j O_j P_j$ and $O$ are equivalent operators. Respect to equation (9), the projection operator $P_i$ transforms as a similarity transformation under the action of operator $\sum_j O_j P_j$, therefor the initial basis should be transformed by this operator and consequently $\sum_j O_j P_j$ and $O$ are equivalent.
We show the identity (4) is also valid in function space, where the linear projection operators are defined.
**Proposition 2.3** Let $\mathbb{V}$ be a n-dimensional function space over the real field $F$ with a set of orthogonal basis functions $\varphi_i$ and inner product defined on a closed interval $[a, b]$.The same definition in (2) can be applied on these basis

$$P_i F = c_i \varphi_i = \varphi_i \int_a^b \varphi_i F \, dx \tag{13}$$

Where
$$c_i = \langle \varphi_i \, , F \rangle = \int_a^b \varphi_i F \, dx \tag{14}$$

Are the coefficients in expansion of square integrable function $F$ in the basis $\varphi_i$ calculated by inner product of $\varphi_i$ and $F$ over the interval $[a, b]$.

**Proof**: with the identity (4) we conclude:

$$P_i'(\sum_j O_j P_j) = O_i P_i$$

Then we have:
$$P_i'(\sum_j O_j P_j) F = O_i P_i F$$

Respect to (13) and (14) we have:

$$P_i'(\sum_j O_j \, \varphi_j \int \varphi_j F \, dx) = O_i \varphi_i \int_a^b \varphi_i F \, dx \tag{15}$$

Regarding (1) we can choose $\varphi'_i = O_i \varphi_i$ as transformed basis that results in:

$$P_i'(\sum_j \varphi'_j \int_a^b \varphi_j F \, dx) = \varphi'_i \int_a^b \varphi_i F \, dx \tag{16}$$

With the definition (2) of projection operator we have:

$$P_i' \varphi'_j = \delta_{ij} \, \varphi'_j$$

Therefor the equation (16) respect to (14) reads as:

$$\varphi'_i \int_a^b \varphi_i F \, dx = c_i \varphi'_i$$

$$c_i \varphi'_i = c_i \varphi'_i$$

So, the identity (9) is valid for operators in function spaces.

**Proposition 2.4** Differential operators with certain eigenvalues and eigenfunctions can be linearly expanded by their projection operators.

**Proof:** Let the differential operator $\mathfrak{D}$ is characterized by eigenfunctions relation:

$$\mathfrak{D} \varphi_i = \lambda_i \varphi_i \tag{17}$$

The eigenfunctions $\varphi_i$ are linearly independent and are the basis vectors, i.e.: $P_i \varphi_j = \delta_{ij} \, \varphi_j$ . Where $P_i$ is the projection on *i*-th subspace, then by the identity:

$$\mathfrak{D} \varphi_i = \lambda_i \varphi_i = (\sum_j \lambda_j P_j) \, \varphi_i \tag{18}$$

The validity of this equation for all $\varphi_i$ yields:

$$\mathfrak{D} = \sum_j \lambda_j P_j \tag{19}$$

That proves the proposition.

**Theorem 2.2** Let the initial basis $e_i$ correspond to some set of linearly independent non-homogenous polynomials such as the regular bases $(1, x, x^2, x^3, \dots)$. After transforming the bases by equation $O_i e_i = e'_i$ to new bases $e'_i$ which correspond the new linearly independent polynomials $\mathrm{P}_n(x)$, if $\mathfrak{D}$ denoted as the differential operator with $e_i$ or equivalently $x^n$ (n-th exponent of *x*) as its eigenfunctions (or eigenvector), then the corresponding differential operator $\mathfrak{D}'$ with eigenfunctions $\mathrm{P}_n(x)$ can be obtained by the relation:

$$\mathfrak{D}' = (\sum_k \lambda'_k O_k P_k)(\sum_j O_j P_j)^{-1} \tag{20}$$

Where $\lambda'_k$ are eigenvalues of $\mathfrak{D}'$.

**Proof** Respect to equation (19) the expansion of $\mathfrak{D}'$ in terms of $P_i{}'$ reads as:

$$\mathfrak{D}' = \sum_i \lambda'_i P_i{}' \tag{21}$$

Where $P_j{}'$ are projection operators onto the i-th subspace ( i.e., $e'_i$). Substitution of $P_i{}'$ in equation (21) by equation (4) results in:

$$\mathfrak{D}' = \sum_i \lambda'_i O_i P_i (\sum_j O_j P_j)^{-1} = (\sum_i \lambda'_i O_i P_i)(\sum_j O_j P_j)^{-1} \qquad (22)$$

This proves the theorem.

**Projection operators in terms of resolvents:**

Associated to any differential operator in Hilbert space there are projection operators in terms of their resolvent i.e. $P_i = \int_{c_{v_i}} \frac{d\lambda}{(\lambda I - \mathfrak{D})^{-1}}$ and $P'_i = \int_{c_{v'_i}} \frac{d\lambda}{(\lambda I - \mathfrak{D}')^{-1}}$ (23)

Using (21), (22) we obtain:

$$\int_{c_{v'_i}} \frac{d\mu}{\mu I - \mathfrak{D}'} = ( O_i \int_{c_{v_i}} \frac{d\lambda}{\lambda I - \mathfrak{D}})(\sum_j O_j \int_{c_{v_j}} \frac{d\lambda}{\lambda I - \mathfrak{D}})^{-1} \qquad (24)$$

For a unique transformation $O = O_i$ for all $\varphi_i$we get:

$$\int_{c_{v'_i}} \frac{d\mu}{\mu I - \mathfrak{D}'} = O(\int_{c_{v_i}} \frac{d\lambda}{\lambda I - \mathfrak{D}})O^{-1}$$

with expansion of resolvents $\frac{d\lambda}{\lambda I - \mathfrak{D}}$ as a Neumann infinite series (polynomial), it is proved that the corresponding differential operator after action of operator $O$ on the base functions $\varphi_i$ as defined in proposition 2.1, can be presented by a similarity transformation:

$$\mathfrak{D}' = O\mathfrak{D}O^{-1} \qquad (25)$$

**Example 2.1**: Eigenfunctions of the differential operator $\mathfrak{D} = \frac{d}{dx}$ could be found as $\varphi_n = e^{nx}$. Transforming by $\varphi'_n = O\varphi_n = x\varphi_n = xe^{nx}$ , The resulting corresponding differential operator respect to proposition 2.1 after substituting $O = x$ reads as:

$$\mathfrak{D}' = x\mathfrak{D}x^{-1} = x\left(\frac{-1}{x^2} + x^{-1}\mathfrak{D}\right) = \frac{-1}{x} + \mathfrak{D} \qquad (26)$$

Action of this operator on $xe^{nx}$ gives:

$$(\frac{-1}{x} + \mathfrak{D})\, xe^{nx} = -e^{nx} + e^{nx} + nxe^{nx} = nxe^{nx} \qquad (27)$$

Thus, the eigenfunctions of this operator are $xe^{nx}$ as expected. Because of the similarity relations of operators $\mathfrak{D}$ and $\mathfrak{D}'$, their eigenvalues are identical.

**Theorem 2.3** Let the linearly independent monomials $p_m(x)$ and $q_m(x)$ of polynomials $\mathrm{P}_n(x)$ and $\mathrm{Q}_n(x)$ of degree $n$ are connected by the operators $O_i$ as defined in equation (1) i.e.,

$$q_m(x) = O_m p_m(x) \qquad (28)$$

Denote $P_m$ as projection operators that project functions of variable $x$ on basis $p_m(x)$ with the definition of equation (2) $P_m p_n(x) = \delta_{mn}\, p_n(x)$

Then the operator $O = \sum_m O_m P_m$ acts as *umbral* composition on polynomial $\mathrm{P}_n(x)$.

**Proof:** Let expand $\mathrm{P}_n(x)$ in terms of monomial basis $p_m(x)$

$$\mathrm{P}_n(x) = \sum_n a_{nm} p_m(x) \qquad (29)$$

Then action of $O$ on $\mathrm{P}_n(x)$ gives

$$O\mathrm{P}_n(x) = \sum_i O_i P_i (\sum_m a_{nm} p_m(x)) = \sum_i O_i\, a_{ni} p_i(x) = \sum_i a_{ni} q_i(x) \qquad (30)$$

This implies that the action of $O$ on $\mathrm{P}_n(x)$ replaces the monomials $p_m(x)$ with $q_i(x)$ while the coefficients $a_{nm}$ in the expansion remains unchanged. This means that the operation of $O$ is equivalent with umbral composition by the definition

$$\mathrm{P}_n \text{ o } \mathrm{Q} = \sum_m a_{nm} q_m(x) \quad (31)$$

This definition coincides the action of $O$ on $\mathrm{P}_n(x)$. Application of this theorem for finding the generating function of Hermite polynomials has been shown in section 3.1.

**Forbenius covariant of operators**

For other representation of projection operator in terms of differential operator we apply the Forbenius covariants [7] as projection operators (matrices) which are the coefficient of *Sylvester's* formula. For a differential operator $\mathfrak{D}$ in polynomial space, the projection operator on the one-dimensional eigenfunction subspaces are given by

$$P_l = \prod_{k=1}^{n} \frac{\mathfrak{D} - \lambda_k}{\lambda_l - \lambda_k} \qquad k \neq l \quad (32)$$

These operators act on the functions in function space and yields their projections on basis $\varphi_i$ which are the eigenfunctions of $\mathfrak{D}$ with corresponding eigenvalues $\lambda_i$.

**Similarity transformation:**

Respect to equation (9) and related proposition, if we substitute $O_i P_i$ with $(\sum_k O_k P_k)$ respect to the identity, $P_i P_j = \delta_{ij} P_i$ we have:

$$P_i' = (\sum_k O_k P_k) P_i (\sum_j O_j P_j)^{-1} = O_i P_i (\sum_j O_j P_j)^{-1} \quad (33)$$

This equation is a similarity transformation of $P_i$ under the operator $\sum_k O_k P_k$. This similarity transformation corresponds to the basis transformation $O_i e_i = e'_i$. Actually, $\sum_k O_k P_k$ as an operator $\hat{O}$ transforms all basis $e_i$ to $e'_i$ and corresponds the coordinate transformation matrix. From this equation we can deduce similarity transformation for other operators provided that the operators in similarity transformation have common eigenvalues. Therefor the differential operators with *identical eigenvalues* could be related by similarity transformation. As an example, differential operator $\mathfrak{D} = x\frac{d}{dx} = xD$ with basis (eigenfunction) $\varphi_n = x^n$ transforms to another differential operator $\mathfrak{D}'$ with eigenfunction $\varphi'_i$ after the basis transformation $\varphi'_i = O_i \varphi_i$. Therefor we have the similarity transformation:

$$\mathfrak{D}' = (\sum_k O_k P_k) \mathfrak{D} (\sum_j O_j P_j)^{-1} \quad (34)$$

If all $O_j$ are the same namely $O$, (34) will be reduced to:

$$\mathfrak{D}' = (O \sum_k P_k) \mathfrak{D} (O \sum_j P_j)^{-1}$$

$$\mathfrak{D}' = O \mathfrak{D} O^{-1}$$

In these cases that the single operator transforms all bases, the exact closed form of related differential operator could be derived by this method. However, for cases with separate $O_i$, the validity of the retrieved differential operator relies on the action on the first two polynomials as we show in next sections. The following example clarifies the method.

**Example 2.2**:

Let the vector space $\mathbb{V}$ spanned by the linearly independent basis $(1, e^x, e^{2x}, \ \ldots)$ which are the eigenfunctions of operator $\mathfrak{D} = \frac{d}{dx}$. If these basis transforms to the new set of basis by multiplying with $e^{\frac{x^2}{2}}$ i.e., $(e^{\frac{x^2}{2}}, e^{\frac{x^2}{2}+x}, e^{\frac{x^2}{2}+2x}, \ \ldots)$ then the corresponding operator with these new basis as its eigenfunctions could be obtained by (34). In this case $O_k = O = e^{\frac{x^2}{2}}$. Thus, the equation (34) reduces to:

$$\mathfrak{D}' = O\mathfrak{D}O^{-1}$$
$$\mathfrak{D}' = e^{\frac{x^2}{2}}\,\mathfrak{D}\,e^{\frac{-x^2}{2}}$$

The term $\mathfrak{D}\,e^{\frac{-x^2}{2}}$ is not just the derivative of $e^{\frac{-x^2}{2}}$, but an operator that is equal to:

$$\mathfrak{D}\,e^{\frac{-x^2}{2}} = \frac{d}{dx}\left(e^{\frac{-x^2}{2}}\right) + e^{\frac{-x^2}{2}}\mathfrak{D}$$

Then we have:

$$\mathfrak{D}' = e^{\frac{x^2}{2}}\,\mathfrak{D}\,e^{\frac{-x^2}{2}} = e^{\frac{x^2}{2}}\left[\frac{d}{dx}\left(e^{\frac{-x^2}{2}}\right) + e^{\frac{-x^2}{2}}\mathfrak{D}\right]$$
$$\mathfrak{D}' = e^{\frac{x^2}{2}}\left(-xe^{\frac{-x^2}{2}} + e^{\frac{-x^2}{2}}\mathfrak{D}\right)$$
$$\mathfrak{D}' = (-x + \mathfrak{D})$$

The eigenfunctions of this operator are $e^{\frac{x^2}{2}+nx}$ with eigenvalues $n$ as expected. It is noteworthy to note that the expression for probabilist's Hermite polynomial $H_{e1}$ with the definition:

$$H_{e1} = e^{\frac{x^2}{2}}\,\frac{d}{dx}\,e^{\frac{-x^2}{2}}$$

Differs from $\mathfrak{D}'$, because in this definition the term $\frac{d}{dx}\,e^{\frac{-x^2}{2}}$ is not an operator but merely the derivative of $e^{\frac{-x^2}{2}}$.

It is easy to prove that any function of $\mathfrak{D}'$ can be expanded in terms of $\mathfrak{D}$ as follows:

$$f(\mathfrak{D}') = Of(\mathfrak{D})O^{-1}$$

**Separated basis transformation method based on Forbenius covariants**

Another approach to find $\mathfrak{D}'$ in terms of $\mathfrak{D}$ and $O_j$ is to apply the Forbenius covariant operators as projection operators as mentioned in (32).

$$P_k = \prod_{l=1}^{N} \frac{\mathfrak{D}-\lambda_l}{\lambda_k-\lambda_l} \qquad l \neq k \tag{35}$$

These operators are projection operators onto the k-th one-dimensional sub-space (basis) [8]. substituting these projectors in equation (34) results in

$$\mathfrak{D}' = \left(\sum_k O_k \prod_{l=1}^{N} \frac{\mathfrak{D}-\lambda_l}{\lambda_k-\lambda_l}\right)\mathfrak{D}\left(\sum_j O_j \prod_{l=1}^{N} \frac{\mathfrak{D}-\lambda_l}{\lambda_j-\lambda_l}\right)^{-1} \qquad l \neq j \tag{36}$$

$N$ denoted as the dimension of function or polynomial space.

This method in comparison with previous methods are more applicable because the calculation of inverse of a product of differential operators is easier than other methods.

In the following sections we introduce an applicable method to find $\mathfrak{D}'$ in terms of $\mathfrak{D}$. Taking into account the equation (22) we have:

$$\mathfrak{D}' = \left(\sum_i \lambda'_i O_i P_i\right)\left(\sum_j O_j P_j\right)^{-1} \tag{37}$$

If all $O_i$ are the same i.e., $O_i = O$, then (37) reduces to

$$\mathfrak{D}' = O\left(\sum_i \lambda'_i P_i\right)O^{-1} \tag{38}$$

The condition of identical eigenvalues for $\mathfrak{D}'$ and $\mathfrak{D}$ is not necessary in equation (37) and the case of identical eigenvalues are special case of this equation. we apply this equation restricted to the first two polynomials i.e., two-dimensional polynomial space. Substitution of $P_i$ in (37) by Forbenius covariants (35) yields an applicable method as we will show in examples. It is noteworthy to recall that the term $\sum_j O_j P_j$ stands for a linear operator (equivalent to a matrix) that transforms the basis$(1, x, x^2, \ldots)$ of polynomials space to another basis. For example, it transforms basis $(1, x, x^2, \ldots)$ to Hermite polynomial $H_{en}$ as new linearly independent basis by the techniques that is presented in next section.

**Applications of separated basis transformation method**

In the Sturm Liouville problem and related differential equations and their specific solutions such as Hermite, Laguerre, Legendre and Jacobi polynomials, the transformation of basis in function space seems to be an interesting subject. For example, transformation of basis $(1, x, x^2, x^3, \ldots)$ under the multiple differentiation which is compatible with Rodrigues' formula to derive Hermite polynomials, presented as follows:

$$H_{en} = e^{\frac{-D^2}{2}} x^n \tag{39}$$

Where $D = \frac{d}{dx}$.

In the case of Laguerre polynomials, we have the transformation:

$$L_n = \frac{1}{n!}(D-1)^n x^n \tag{40}$$

Respect to our theory, these transformations are compatible with the operator action of $O_n$ separately on basis $(1, x, x^2, x^3, \ldots)$, for Hermite polynomial we have:

$$O_n = O = e^{\frac{-D^2}{2}} \tag{41}$$

And for Laguerre polynomials:

$$O_n = \frac{1}{n!}(D-1)^n \tag{42}$$

We introduce the operator $xD$ as the unique operator with basis $(1, x, x^2, x^3, \ldots)$ as its associated eigenfunctions with eigenvalues $(0,1,2, \ldots)$:

$$xD\,(x^n) = nx^n$$

Therefor we can use the equation (38) to find the differential operator that its polynomials are determined by applying related $O_n$ on basis $(1, x, x^2, x^3, \ldots)$ as in (39) and (40). By substitution of $P_k$ and $O_n$ in equations (36) and (37) and the Forbenius covariants (35) and $\mathfrak{D} = xD$ in equation (36) we recover the corresponding differential equations of eigenfunctions such as $H_{en}$ and $L_n$ as presented in net examples. The presented technique uses the first two polynomials i.e., the two-dimensional space of polynomials with monomials of order 1 and 0. This facilitates the calculation of desired differential equations and shows that if an infinite set of polynomials present the eigenfunctions of a unique differential operator, then applying this technique for the first two polynomials gives the exact form of related differential equation. We clarify this method by the following proposition

**Proposition 2.5** Let the set of linearly independent polynomials $\mathbb{P}_n$ are the eigenfunctions of a differential operator $\mathfrak{D}'$ and the set of original basis $[1, B(x), B^2(x), \ldots, B^n(x)]$ are the eigenfunctions of differential operator $\mathfrak{D}$. Then applying the Forbenius covariant operator defined in (35) and equations (36) and (37) for the first two polynomials (eigenfunctions) $\mathbb{P}_0$ and $\mathbb{P}_1$, yields the corresponding differential operator $\mathfrak{D}'$ from $\mathfrak{D}$.

First, we prove this proposition for Rodrigues formula as the action of operators $O_n$ on the initial basis $B^n(x)$ in polynomial space to transform them to new basis $\mathbb{P}_n$ that correspond to the desired differential operator $\mathfrak{D}'$(i.e., differential equation) as its eigenfunctions.

**Rodrigues' formula as a special case of separated basis transformation**

In this section we prove the compatibility of Rodrigues' formula with our presented techniques and show that substitution of $O_n$ in equation (43) by Rodrigues' formula transformation, yields the corresponding differential operators and equations.

**Proof**

Due to the presented theory, we showed that if the bases $e_n$ of a vector space $\mathbb{V}$ which are the eigenfunctions of differential operator $\mathfrak{D}$, are transformed separately by operators $O_n$, the transformed differential operator obeys the equation (43) i.e.,

$$\mathfrak{D}' = (\sum_i \lambda'_i O_i P_i)(\sum_j O_j P_j)^{-1} \tag{43}$$

We check the basis transformation by Rodrigues formula [3]:

$$\mathbb{P}_n = \frac{1}{\omega} D^n[\omega B^n(x)]$$

Where $\omega$ defined by the relation $\frac{\omega}{\omega'} = \frac{A-B'}{B}$ with $A$ as a polynomial of first degree.

If we choose monomial $B^n(x)$ as the original basis of vector space:

$$[1 , B(x),\ B^2(x) , \ldots , B^n(x)]$$

The Rodrigues formula could be chosen as the action of operator:

$$O_n = \frac{1}{\omega} D^n[\omega\ .] \tag{44}$$

On these basis. Therefor it is a special case of separated basis transformation.

The suitable operator with eigenfunctions $B^n(x)$ can be presented as

$$\frac{B(x)}{B'(x)} D\ B^n(x) = nB^n(x) \tag{45}$$

Where $B'(x)$ denoted as the derivative of $B(x)$ The $\frac{B(x)}{B'(x)} D$ should replace $\mathfrak{D}$ in (35):

$$P_k = \prod_{l=1}^{N} \frac{\frac{B(x)}{B'(x)} D - \lambda_l}{\lambda_k - \lambda_l} \tag{46}$$

Respect to (43) by replacing $O_n$ by Rodrigues formula and $P_n$ by equation (46) we get:

$$\mathfrak{D}' = (\sum_i \lambda'_i \frac{1}{\omega} D^n[\omega \prod_{l=1}^{N} \frac{\frac{B(x)}{B'(x)} D - \lambda_l}{\lambda_k - \lambda_l}])(\sum_j O_j P_j)^{-1}$$

Let $O^{-1} = (\sum_j O_j P_j)^{-1}$ then we obtain:

$$\mathfrak{D}' = (\sum_i \lambda'_i \frac{1}{\omega} D^n[\omega \prod_{l=1}^{N} \frac{\frac{B(x)}{B'(x)} D - \lambda_l}{\lambda_k - \lambda_l}])O^{-1} \tag{47}$$

Taking into account the 2-dimensional space, and using (44) and (47) we have:

$$\lambda'_0 = 0 \quad , \quad O_1 = \frac{1}{\omega} D[\omega\ .] \quad , \quad P_1 = \frac{\frac{B(x)}{B'(x)} D}{\lambda'_1} = \frac{1}{\lambda'_1} \frac{B(x)}{B'(x)} D$$

Thus the (47) reads as:

$$\mathfrak{D}' = \lambda'_1 \frac{1}{\omega} D\left[\omega \frac{1}{\lambda'_1} \frac{B(x)}{B'(x)} D\right] O^{-1} = \frac{1}{\omega} D\left[\omega \frac{B(x)}{B'(x)} D\right] O^{-1} \tag{48}$$

$$\mathfrak{D}' = (\frac{\omega}{\omega'}\frac{B}{B'}D + \frac{B'^2 - B''B}{B'^2}D + \frac{B}{B'}D^2)O^{-1}$$

If we assume $\frac{\omega}{\omega'} = \frac{A-B'}{B}$ ( as a crucial assumption in Rodrigues formula) this equation reduces to:

$$\mathfrak{D}' = \left\{\left(\frac{A}{B'} - 1\right)D + D - \frac{B''B}{B'^2}D + \frac{B}{B'}D^2\right\}O^{-1}$$

$$\mathfrak{D}' = (\frac{A}{B'}D - \frac{B''B}{B'^2}D + \frac{B}{B'}D^2)O^{-1} \tag{49}$$

Acting both side on $\mathbb{P}_1$ as the second eigenfunction of $\mathfrak{D}'$, we have:

$$\mathfrak{D}'\mathbb{P}_1 = (\frac{A}{B'}D - \frac{B''B}{B'^2}D + \frac{B}{B'}D^2)O^{-1}\mathbb{P}_1 \tag{50}$$

The term $O^{-1}\mathbb{P}_1$ equals $B(x)$, thus:

$$\mathfrak{D}'\mathbb{P}_1 = (\frac{A}{B'}D - \frac{B''B}{B'^2}D + \frac{B}{B'}D^2)B(x) \tag{51}$$

$$\mathfrak{D}'\mathbb{P}_1 = \left(\frac{A}{B'} - \frac{B''B}{B'^2} + \frac{B}{B'}D\right)B' = A - \frac{B''B}{B'} + \frac{B}{B'}(B'' + B'D)$$

$$\mathfrak{D}'\mathbb{P}_1 = BD + A$$

$$\mathfrak{D}'D^{-1}D\mathbb{P}_1 = BD + A \tag{52}$$

The term $D\mathbb{P}_1$will be a constant $\alpha$, thus:

$$\alpha\mathfrak{D}'D^{-1} = BD + A$$

Or:

$$\alpha\mathfrak{D}' = BD^2 + AD \tag{53}$$

This implies that Rodrigues formula gives the solutions (or eigenfunctions) of the differential operator $BD^2 + AD$ and related differential equation up to a constant coefficient $\alpha$. i.e.,

$$(BD^2 + AD)y = \beta y$$

The following examples clarify this technique for some polynomials.

**Example 2.3: Laguerre differential equation**

Let we intend to find the differential equation which corresponds to a set of linearly independent polynomials in variable $x$. For example, we are given a few first Laguerre polynomials i.e., $(1, 1 - x, \dots)$ and we know the operator that maps the standard basis $(1, x, x^2, \dots)$ to Laguerre basis i.e., operator presented in (42).

We can recover the corresponding Laguerre differential equation (operator) via the formula:

$$\mathfrak{D}' = (\sum_i \lambda'_i O_i P_i)(\sum_j O_j P_j)^{-1} \tag{54}$$

*Proof in 2 dimension (first 2 polynomials)*

We restrict calculation in 2-dimensional polynomial space with basis $(1, x)$.These polynomials are transformed by $O_i$ to Laguerre polynomials in the same dimension i.e., $(1, -x + 1)$. Thus, the corresponding operator $xD$ will be transformed to Laguerre differential operator by the equation (43).

Substitution of $O_i$ by equation (42) and taking $\lambda'_i$ as the eigenvalues of Laguerre differential equation in 2-dimensional space of polynomials and replacing projection operators $P_i$ for basis $(1, x)$ by equation (35) into equation (37) results in:

$$\mathfrak{D}' = (\sum_{i=0}^{1} \lambda'_i O_i P_i)(\sum_{j=0}^{1} O_j P_j)^{-1} \tag{55}$$

We have $\lambda'_i = \lambda_i$ and $\lambda_0 = 0$ , $\lambda_1 = 1$. Then equation (44) reduces to:

$$\mathfrak{D}' = O_1 P_1 (O_0 P_0 + O_1 P_1)^{-1} \tag{56}$$

By equations (35) and (42) we obtain:

$$O_0 = 1 \quad , \quad O_1 = D-1 \; , \; P_0 = \prod_{l=0}^{1} \frac{\mathfrak{D}-\lambda_1}{\lambda_0-\lambda_1} = \frac{xD-1}{-1} \quad , \quad P_1 = \prod_{l=0}^{1} \frac{\mathfrak{D}-\lambda_0}{\lambda_1-\lambda_0} = \frac{xD}{1}$$

Therefor we have:

$$O_1P_1 = (D-1)xD = D + xD^2 - xD$$

And: $$\mathfrak{D}' = O_1P_1(O_0P_0 + O_1P_1)^{-1} = (D + xD^2 - xD\ )(\ D + xD^2 - 2xD + 1)^{-1} \tag{57}$$

If we denote the $(D + xD^2 - xD)$ as $\mathbb{D}$, we can reduce the equation (46) as follows:

$$\mathfrak{D}' = O_1P_1(O_0P_0 + O_1P_1)^{-1} = \mathbb{D}\,(\,\mathbb{D} - \mathfrak{D} + 1)^{-1} \tag{58}$$

The term

$$O_0P_0 + O_1P_1 = \mathbb{D} - \mathfrak{D} + 1 = \hat{O} \tag{59}$$

Is the linear operator which transforms the basis $(1, x)$ to Laguerre basis $(1, -x+1)$ and vice versa If we restrict the action of operators to 2-dimensional polynomial space. Therefor we have:

$$\hat{O}^{-1}(-x+1) = x \tag{60}$$

If we act both sides of (47) on another basis $(-x+1)$ we get as well:

$$\mathfrak{D}'(-x+1) = \mathbb{D}\hat{O}^{-1}(-x+1) = \mathbb{D}x = -\mathbb{D}(-x+1)$$

Or briefly: $$\mathfrak{D}'(-x+1) = -\mathbb{D}(-x+1) \tag{61}$$

(Note that $-\mathbb{D}.1 = 0$)

The equation (61) implies that the action of both operators $\mathfrak{D}'$ and $-\mathbb{D}$ on basis $(1, -x+1)$ are identical and therefor the simplest form of operator $\mathfrak{D}'$ which its eigenfunctions are Laguerre polynomials and its related transformation operators are $O_i$, reads as:

$$\mathfrak{D}' = -\mathbb{D} = -(xD^2 - xD + D) \tag{62}$$

This is the exactly the Laguerre differential equation with positive eigenvalues, i.e.:

$$-(xD^2 - xD + D)y = ny \tag{63}$$

Action of this operator on the first basis i.e., “1” gives 0 as the first eigenvalue and therefor the required conditions for validity of this differential operator are met.

*Proof in 3 dimension (first 3 polynomials)*

In 3-dimension with basis $(1, -x+1, \frac{1}{2}(x^2 - 4x + 2)$ of Laguerre polynomial and $(1, x, x^2)$ of original basis, considering eigenvalues $\lambda_0 = 0\,, \lambda_1 = 1\,, \lambda_2 = 2,$ the $\mathfrak{D}'$ reads as:

$$\mathfrak{D}' = \left(\sum_i \lambda'_i O_i P_i\right)\left(\sum_j O_j P_j\right)^{-1} = (O_1P_1 + 2O_2P_2)(O_0P_0 + O_1P_1 + O_2P_2)^{-1} \tag{64}$$

$$\mathfrak{D}' = (O_1P_1 + 2O_2P_2)\ \hat{O}^{-1}$$

Here $\hat{O}^{-1}$ denotes the last term in (64). Acting both side on basis $(-x+1)$ results in:

$$\mathfrak{D}'(-x+1) \ = (O_1P_1 + 2O_2P_2)\ \hat{O}^{-1}(-x+1)$$

Respect to (60) and the identity $P_2x = 0$ we have:

$$\mathfrak{D}'(-x+1) = O_1P_1\ x \tag{65}$$

In this dimension $P_1$ can be find as:

$$P_1 = \prod_{l=0}^{2} \frac{\mathfrak{D}-\lambda_l}{\lambda_1-\lambda_l} = \left(\frac{xD-\lambda_0}{1-\lambda_0}\right)\left(\frac{xD-\lambda_2}{1-\lambda_2}\right) = xD\left(\frac{xD-2}{1-2}\right) = -xD(xD-2)$$

Then (65) reads as:

$$\mathfrak{D}'(-x+1) = -(D-1)xD(xD-2)\ x$$

$$\mathfrak{D}'(-x+1) = -(D-1)xD(-x)$$

$$\mathfrak{D}'(-x+1) = -[(D-1)xD](-x+1)$$
$$\mathfrak{D}'(-x+1) = -(xD^2 - xD + D)(-x+1)$$
This proves:
$$\mathfrak{D}' = -(xD^2 - xD + D)$$
As the Laguerre differential operator.

**Example 2.4: Hermite differential equation**

The same technique could be applied to derive Hermite differential equation by the formula (37). Because all $O_n$ that transforms basis $(1, x, x^2, x^3, \dots)$ to Hermite polynomials are equal to $O$ as is shown in (41), after getting $P_k$ by (35) and substitute them in (37) we have:

$$\begin{aligned}\mathfrak{D}' &= (\textstyle\sum_i \lambda'_i O P_i)(\sum_j O P_j)^{-1}\\ &= O(\textstyle\sum_i \lambda'_i P_i)(O \sum_j P_j)^{-1}\\ &= O(\textstyle\sum_i \lambda'_i P_i)(\sum_j P_j)^{-1} O^{-1}\end{aligned}$$
Respect to $\sum_j P_j = 1$ we get:
$$\mathfrak{D}' = O(\textstyle\sum_i \lambda'_i P_i)O^{-1}$$
Expanding the sum for eigenvalues $\lambda'_i = \lambda_i = 0{,}1$ and substitution of $O$ by (41)we have:
$$\mathfrak{D}' = e^{\frac{-D^2}{2}}(P_1)e^{\frac{D^2}{2}}$$
From (35) we calculate $P_1$ as:
$$P_1 = \textstyle\prod_{l=0}^{1} \frac{\mathfrak{D}-\lambda_0}{\lambda_1-\lambda_0} = \frac{\mathfrak{D}-0}{1-0} = \mathfrak{D}$$
We know $(1, x, x^2, x^3, \dots)$ are the eigenfunctions of $xD$, thus by $\mathfrak{D} = xD$ we have:
$$\mathfrak{D}' = e^{\frac{-D^2}{2}}(xD)e^{\frac{D^2}{2}} \tag{66}$$
This equation can be interpreted as a similarity transformation that maps $xD$ into $\mathfrak{D}'$ after basis changes. This will be hold just for the cases that eigenvalues are common between $xD$ and $\mathfrak{D}'$ as we see in Hermite and Laguerre differential equations.

Expansion of $e^{\frac{-D^2}{2}}$ and $e^{\frac{D^2}{2}}$ results in:
$$\mathfrak{D}' = (1 - \frac{D^2}{2} + \frac{D^4}{8} + \cdots)(xD)(1 + \frac{D^2}{2} + \frac{D^4}{8} + \cdots) \tag{67}$$
$$\mathfrak{D}' = (1 - \frac{D^2}{2} + \frac{D^4}{8} + \cdots)(xD + x\frac{D^3}{2} + x\frac{D^5}{8} + \cdots)$$
$$\mathfrak{D}' = \left(xD - \frac{D^2}{2}xD + \frac{D^4}{8}xD + \cdots\right) + (1 - \frac{D^2}{2} + \frac{D^4}{8} + \cdots)(x\frac{D^3}{2} + x\frac{D^5}{8} + \cdots)$$
In the 2-dimensional space of polynomials the orders higher than 2 for $D^n$ will be omitted, as it could be verified by action of both side on basis $x$ . By omitting the higher orders, we obtain:
$$\mathfrak{D}' = (1 - \frac{D^2}{2})xD$$
$$\mathfrak{D}' = xD - \frac{D^2}{2}xD = xD - \frac{1}{2}D(D + xD^2)$$
$$\mathfrak{D}' = xD - \frac{1}{2}(D^2 + D^2 + xD^3)$$
Omitting $xD^3$ results in:
$$\mathfrak{D}' = xD - D^2 = -(D^2 - xD) \tag{68}$$
This is the well-known Hermit probabilist's Hermite differential operator with Hermite polynomial as its eigenfunctions and positive eigenvalues 0 , 1 , 2 , …. as its eigenvalues.

**Example 2.5: Legendre differential equation**

For Legendre polynomials we have:

$$\mathbb{P}_n = \frac{1}{2^n n!} D^n (x^2-1)^n \tag{69}$$

That transforms the basis set $S = \{1, (x^2-1), (x^2-1)^2, \ ...\}$ to Legendre polynomials. We can choose the appropriate operator $\mathfrak{D}$ whose eigenfunctions are these basis. Simply we write:

$$\mathfrak{D} = \frac{x^2-1}{2x} D \tag{70}$$

Eigenfunctions of this operator are members of the set $S$.
The transforming operator is

$$O_n = \frac{1}{2^n n!} D^n \tag{71}$$

In this case the eigenvalues of $\mathfrak{D}$ and $\mathfrak{D}'$ (Legendre differential operator) are not identical and therefor the similarity transformation is not valid. However, we can apply the equation (37) after determining the $P_i$ from (35).
For calculating $P_i$ in 2-dimension, we have:

$$P_0 = \prod_{l=0}^{1} \frac{\mathfrak{D}-\lambda_1}{\lambda_0-\lambda_1} = \frac{\frac{x^2-1}{2x}D-1}{-1} = 1 - \frac{x^2-1}{2x} D \tag{72}$$

$$P_1 = \prod_{l=0}^{1} \frac{\mathfrak{D}-\lambda_0}{\lambda_1-\lambda_0} = \frac{\frac{x^2-1}{2x}D-0}{1-0} = \frac{x^2-1}{2x} D \tag{73}$$

Noe we get:

$$\sum_i \lambda'_i O_i P_i = 2O_1 \frac{x^2-1}{2x} D = 2\left(\frac{1}{2} D \frac{x^2-1}{2x} D\right) = D \frac{x^2-1}{2x} D \tag{74}$$

From equation (37) and (63) we get:

$$\mathfrak{D}' = \left(\sum_i \lambda'_i O_i P_i\right)\left(\sum_j O_j P_j\right)^{-1}$$

$$\mathfrak{D}' = D \frac{x^2-1}{2x} D \left(\sum_j O_j P_j\right)^{-1} = D \frac{x^2-1}{2x} D O^{-1} \tag{75}$$

Where $O^{-1} = \left(\sum_j O_j P_j\right)^{-1}$.
By action of both sides of (75) on basis $x$ as the second basis of Legendre polynomials in two dimension, we have:

$$\mathfrak{D}' x = D \frac{x^2-1}{2x} D O^{-1} x \tag{76}$$

Respect to $O^{-1} x = x^2 - 1$, (65) reads as:

$$\mathfrak{D}' x = D \frac{x^2-1}{2x} D (x^2-1)$$

$$\mathfrak{D}'(D^{-1}D) x = D \frac{x^2-1}{2x} (2x)$$

$$\mathfrak{D}' D^{-1}(Dx) = D(x^2-1)$$

$$\mathfrak{D}' D^{-1} = D(x^2-1)$$

$$\mathfrak{D}' = D(x^2-1)D = -D(1-x^2)D \tag{77}$$

Expansion of (77) reads as:

$$\mathfrak{D}' = -[(1-x^2)D^2 - 2xD]$$

Which is the Legendre differential operator with positive eigenvalues $n(n+1)$.

## 3. Hermit, Laguerre, and Legendre differential operator as Cartan subalgebra of $\mathfrak{sl}(2,R)$ and $\mathfrak{su}(2)$

Let $\mathfrak{gl}\,(\mathbb{V})$ denote the linear transformation that maps vector space $\mathbb{V}$ onto itself. In this section we present isomorphic Lie algebras to $\mathfrak{sl}(2,R)$ defined by $\mathfrak{sl}(2,R)$ module on vector space $\mathbb{V}$ which is a linear map $\phi$ defined by $\phi : \mathfrak{sl}(2,R) \to \mathfrak{gl}\,(\mathbb{V})$ that preserves the commutator relations of $\mathfrak{sl}(2,R)$ algebra [4,8].

$$\phi[a,b] = [\phi(a),\phi(b)] \qquad a,b \in \mathfrak{sl}(2,R)$$

This representation is $\mathfrak{sl}(2,R)$ module on vector space $\mathbb{V}$.

First, we review the structure of irreducible vector field representation of $\mathfrak{sl}(2,R)$.The generators of this algebra in matrix representation are as follows:

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The commutation relations for this representation of $\mathfrak{sl}(2,R)$are:

$$[X,Y\,] = 2H \ , \ [H,X\,] = -X \ , \ [H,Y\,] = Y \tag{78}$$

Let define a representation of $\mathfrak{sl}(2,R)$ as its module on $\mathbb{V}$ that preserves commutation relations by differential operators as its generators:

$$\boldsymbol{h} = xD - \frac{n}{2} \ , \qquad \boldsymbol{e} = D = \partial_x \ , \qquad \boldsymbol{f} = x^2D - nx \tag{79}$$

With the similar commutation relations

$$[\boldsymbol{e},\boldsymbol{f}] = 2\boldsymbol{h} \quad , \quad [\boldsymbol{h},\boldsymbol{e}] = -\boldsymbol{e} \quad , [\boldsymbol{h},\boldsymbol{f}] = \boldsymbol{f}$$

The Cartan sub-algebra $H = \boldsymbol{h}$ produces a decomposition of representation space:

$$\mathbb{V} = \bigoplus \mathbb{V}_j$$

$\mathbb{V}_j$ are the eigenspace (eigenfunction) of generator $\boldsymbol{h}$ as Cartan sub-algebra of $\mathfrak{sl}(2,R)$ and provide the solutions to the related differential equation.

$$\boldsymbol{h}\mathbb{V}_j = j\mathbb{V}_j$$

In present paper the eigenspaces $\mathbb{V}_j$ are one dimensional and coincide the basis of polynomial space. These basis are called *weight vectors*. For a finite dimensional representation there is a *highest weight* $j = n$ that determines the dimension of representation space by $\dim\mathbb{V} = n+1$. As an example, the Cartan subalgebra of $\mathfrak{sl}(2,R)$ can be represented by $\boldsymbol{h} = xD$ with $x^n$ as its *weight vectors* (eigenfunctions) and integer $n$ as eigenvalues. Due to the properties of $\mathfrak{sl}(2,R)$, the operator $\boldsymbol{e}$ acts as *lowering* operator $A^-$ and $\boldsymbol{f}$ as *raising* operator $A^+$. The action of these operator on representation basis (eigenfunction) of $\boldsymbol{h}$ lowers or raise the power of $x^n$.

$$\boldsymbol{e}\mathbb{V}_j = \alpha\mathbb{V}_{j-1} \qquad , \qquad \boldsymbol{f}\mathbb{V}_j = \beta\mathbb{V}_{j+1}$$

In the following sections we will construct a set of isomorphic Lie algebras to $\mathfrak{sl}(2,R)$ based on differential operators of Hermite, Laguerre and Legendre equations whose Cartan sub-algebras are Hermit and Laguerre differential operators. These algebras could be derived by similarity transformations (conjugation) of generators of $\mathfrak{sl}(2,R)$ defined in equation (79). The similarity transformation is achieved by the transforming operator by which the original polynomial space basis transforms to the deemed polynomial i.e., Hermite, Laguerre and Legendre polynomials as transformed basis. These operators could be derived from Rodrigues' formula as has been shown in previous examples. For each algebra there exist a set of lowering and raising operators that derives the recursion equations for related polynomials.

### 3.1 Associated Lie Algebra of Hermite Differential Operator

We search for a Lie algebra $\mathfrak{L}_H$ isomorphic to $\mathfrak{sl}(2,R)$ algebra with generators to be defined based on Hermite differential operators. Here we apply the transformation operator $e^{\frac{-D^2}{2}}$ as described in (41) for Hermite polynomials to derive similarity transformations (conjugation) of $\mathfrak{sl}(2,R)$ bases as follows:

$$X_1 = e^{\frac{-D^2}{2}}\boldsymbol{h}\, e^{\frac{D^2}{2}}\ ,\ X_2 = e^{\frac{-D^2}{2}}\boldsymbol{e}\, e^{\frac{D^2}{2}}\ ,\ X_3 = e^{\frac{-D^2}{2}}\boldsymbol{f}\, e^{\frac{D^2}{2}} \tag{80}$$

Equations (69) are the similarity transformations of Lie algebra $\mathfrak{L}_H$, that results in an algebra with basis $X_i$ isomorphic to $\mathfrak{L}_H$.

Then for $X_1$ we have:

$$X_1 = e^{\frac{-D^2}{2}}\boldsymbol{f}\, e^{\frac{D^2}{2}} \tag{81}$$

$$X_1 = e^{\frac{-D^2}{2}}(xD - \frac{n}{2})e^{\frac{D^2}{2}} \tag{82}$$

Respect to (55) this equation reduces to:

$$X_1 = \mathfrak{D}'_H - \frac{n}{2} \tag{83}$$

Where $\mathfrak{D}'_H = xD - D^2$ as proved in (57), denoted as Hermite differential operator.

For $X_2$ we get:

$$X_2 = e^{\frac{-D^2}{2}}D\, e^{\frac{D^2}{2}}$$

Since the operator $D$ is commutable with both $e^{\frac{-D^2}{2}}$ and $e^{\frac{D^2}{2}}$, we have:

$$X_2 = De^{\frac{-D^2}{2}}\ e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}}\ e^{\frac{D^2}{2}}D = D$$

Similarly, for $X_3$:

$$X_3 = e^{\frac{-D^2}{2}}\ (x^2D - nx\ )e^{\frac{D^2}{2}}$$

$$X_3 = e^{\frac{-D^2}{2}}\ (x^2D)e^{\frac{D^2}{2}} - e^{\frac{-D^2}{2}}(nx)e^{\frac{D^2}{2}}$$

$$X_3 = e^{\frac{-D^2}{2}}\ x^2e^{\frac{D^2}{2}}D - ne^{\frac{-D^2}{2}}\ xe^{\frac{D^2}{2}} \tag{84}$$

To calculate this generator, first we know from (57) that:

$$\mathfrak{D}'_H = e^{\frac{-D^2}{2}}(xD)e^{\frac{D^2}{2}} \tag{85}$$

Because $D$ commutes with $e^{\frac{D^2}{2}}$ we obtain:

$$\mathfrak{D}'_H = e^{\frac{-D^2}{2}}xe^{\frac{D^2}{2}}D$$

Or :

$$\mathfrak{D}'_H D^{-1} = e^{\frac{-D^2}{2}}\ xe^{\frac{D^2}{2}}$$

With:

$$\mathfrak{D}'_H D^{-1} = (xD - D^2)D^{-1} = x - D \tag{86}$$

Therefor we have:

$$x - D = e^{\frac{-D^2}{2}}\ xe^{\frac{D^2}{2}}$$

Multiplying this with itself results in:

$$(x-D)^2 = \left(e^{\frac{-D^2}{2}}xe^{\frac{D^2}{2}}\right)\left(e^{\frac{-D^2}{2}}xe^{\frac{D^2}{2}}\right) = e^{\frac{-D^2}{2}}x^2e^{\frac{D^2}{2}} \tag{87}$$

With substitutions, equation (84) reads as:

$$X_3 = (x-D)^2D - n(x-D) \tag{88}$$

Then the list for generators of this representation of $\mathfrak{sl}(2,R)$ is:

$$X_1 = \mathfrak{D}'_H - \frac{n}{2} \quad , \quad X_2 = D \ , \quad X_3 = (x-D)^2 D - n(x-D)(x-D) = (x-D)(\mathfrak{D}'_H - n)$$

The Cartan subalgebra of this algebra is $X_1 = \mathfrak{D}'_H - \frac{n}{2}$.
Clearly these generators span the Lie algebra $\mathfrak{L}_H$ isomorphic to $\mathfrak{sl}(2,R)$, which is a representation for an isomorphism of $\mathfrak{sl}(2,R)$. The commutation relations can be checked as:

$$[X_1, X_2] = \left(\mathfrak{D}'_H - \frac{n}{2}\right) D - D\left(\mathfrak{D}'_H - \frac{n}{2}\right) = -D = -X_2 \qquad (89)$$

$$[X_2, X_3] = D[(x-D)^2 D - n(x-D)] - [(x-D)^2 D - n(x-D)]D \qquad (90)$$
$$= 2\left(xD - D^2 - \frac{n}{2}\right) = 2X_1$$

For $[X_1, X_3]$, first we note: $X_3 = (x-D)(\mathfrak{D}'_H - n)$, and we use $\mathfrak{D}'_H$ instead $X_1$without any change in commutator result. Thus, we have:

$$[X_1, X_3] = \mathfrak{D}'_H(x-D)(\mathfrak{D}'_H - n) - (x-D)(\mathfrak{D}'_H - n)\mathfrak{D}'_H \qquad (91)$$

Due to the identity: $(\mathfrak{D}'_H - n)\mathfrak{D}'_H = \mathfrak{D}'_H(\mathfrak{D}'_H - n)$

The equation (80) becomes:

$$[X_1, X_3] = \mathfrak{D}'_H(x-D)(\mathfrak{D}'_H - n) - (x-D)\mathfrak{D}'_H(\mathfrak{D}'_H - n)$$
$$[X_1, X_3] = [\mathfrak{D}'_H(x-D) - (x-D)\mathfrak{D}'_H](\mathfrak{D}'_H - n)$$
$$[X_1, X_3] = [\mathfrak{D}'_H(x-D) - (x-D)\mathfrak{D}'_H](\mathfrak{D}'_H - n)$$

Substitution of $\mathfrak{D}'_H$ by $xD - D^2$ gives:

$$[X_1, X_3] = [\mathfrak{D}'_H(x-D) - (x-D)(xD - D^2)](\mathfrak{D}'_H - n)$$

Replacing operator $xD$ with its equivalence $Dx - 1$ results in:

$$[X_1, X_3] = [\mathfrak{D}'_H(x-D) - (x-D)(Dx - 1 - D^2)](\mathfrak{D}'_H - n)$$
$$[X_1, X_3] = [\mathfrak{D}'_H(x-D) + (x-D) - (x-D)D(x-D)](\mathfrak{D}'_H - n)$$
$$[X_1, X_3] = [\mathfrak{D}'_H + 1 - (x-D)D](x-D)(\mathfrak{D}'_H - n)$$
$$[X_1, X_3] = [\mathfrak{D}'_H + 1 - (x-D)D](x-D)(\mathfrak{D}'_H - n)$$
$$[X_1, X_3] = (x-D)(\mathfrak{D}'_H - n) = X_3$$

This proves the isomorphism of the Lie algebra $\mathfrak{L}_H$ with basis $X_1, X_2, X_3$ with $\mathfrak{sl}(2,R)$.

**Lowering and Raising operators of Hermite Polynomials and its Generating function**

In this section we introduce the raising and lowering operators of Hermite polynomials which act on vector space representation of $\mathfrak{sl}(2,R)$. We denote raising and lowering operators as $A^+$and $A^-$ respectively. These operators act on the weight vectors which are eigenfunctions of $X_1$or $\mathfrak{D}'_H$ i.e., the Hermite polynomials $\mathbb{H}_{en}$. As an example, for Lie algebra $\mathfrak{L}_H$ the following relations could be considered.

1) Due to the properties of $\mathfrak{sl}(2,R)$ algebra the generator $X_2$ acts as a lowering operator $A^-$. This implies that:

$$D\mathbb{H}_{e_n} = n\mathbb{H}_{e_{n-1}} \qquad (92)$$

2) Consecutive action of the $X_1$ and $X_2$ generators on the eigenfunction $\mathbb{H}_{en}$ of $X_1$(i.e., the Hermite polynomial of degree $n$) results in lowering of polynomial degree. Respect to (81):

$$X_1 X_2 \mathbb{H}_{e_n} = (\mathfrak{D}'_H - \frac{n}{2}) D\mathbb{H}_{e_n}$$
$$= (\mathfrak{D}'_H - \frac{n}{2})\mathbb{H}_{e_{n-1}}$$
$$= n(\frac{n}{2} - 1)\mathbb{H}_{e_{n-1}} \qquad (93)$$

This means that the operator $X_1X_2$ acts as a lowering (ladder) operator $A^-$ in the subspaces spanned by the Cartan subalgebra $X_1$ of $\mathfrak{L}_H$.

3) The raising operator can be derived from equation (85) and (86):

$$\mathfrak{D}'_H = e^{\frac{-D^2}{2}}(xD)e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}}xe^{\frac{D^2}{2}}D$$

$$\mathfrak{D}'_H D^{-1} = e^{\frac{-D^2}{2}}xe^{\frac{D^2}{2}} \tag{94}$$

If we act the right side of (94) on a Hermite polynomial of degree $n$, respect to equation (41) we get:

$$e^{\frac{-D^2}{2}}xe^{\frac{D^2}{2}}\mathbb{H}_{e_n} = e^{\frac{-D^2}{2}}xO^{-1}\mathbb{H}_{e_n} = e^{\frac{-D^2}{2}}x.x^n \tag{95}$$

$$= e^{\frac{-D^2}{2}}x^{n+1} = Ox^{n+1} = \mathbb{H}_{e_{n+1}}$$

Thus (94) and (95) yields:

$$\mathfrak{D}'_H D^{-1}\mathbb{H}_{e_n} = (xD - D^2)D^{-1}\mathbb{H}_{e_n} = (x-D)\mathbb{H}_{e_n} = \mathbb{H}_{e_{n+1}} \tag{96}$$

Therefor the operator $x - D$ acts as raising operator $A^+$ in the associated vector space spanned by $\mathbb{H}_{e_n}$.

4) If this method be repeated for $X_1X_3$ operator, we have:

$$X_1X_3\mathbb{H}_{e_n} = (\mathfrak{D}'_H - \frac{n}{2})[(x-D)^2D - n(x-D)]\mathbb{H}_{e_n}$$

$$= (\mathfrak{D}'_H - \frac{n}{2})(x-D)^2D\mathbb{H}_{e_n} - n(x-D)\mathbb{H}_{e_n}$$

Taking into account (95) and (96) we deduce:

$$X_1X_3\mathbb{H}_{e_n} = (\mathfrak{D}'_H - \frac{n}{2})(x-D)^2\mathbb{H}_{e_{n-1}} - n\mathbb{H}_{e_{n+1}}$$

$$X_1X_3\mathbb{H}_{e_n} = (\mathfrak{D}'_H - \frac{n}{2})\mathbb{H}_{e_{n+1}} - n\mathbb{H}_{e_{n+1}}$$

$$X_1X_3\mathbb{H}_{e_n} = (\frac{n}{2}+1)\mathbb{H}_{e_{n+1}} - n\mathbb{H}_{e_{n+1}} = (1-\frac{n}{2})\mathbb{H}_{e_{n+1}} \tag{97}$$

Clearly the operator $X_1X_3$ acts as a raising operator $A^+$.

The results of this section can be used to derive recursive relations for Hermits polynomials as follows:

Any combination of operators involved in (92),(93),(95),(96) and (97) results in a recursive relation for Hermite polynomials.

5) The generating function of Hermit polynomial can be derived by a method based on theorem 3.2 as follows.

By expansion of $e^{tx}$ and acting the operator $O$ defined in (41) on it and taking into account the umbral property of $O$ proved in theorem 2.3. We have

$$g(x,t) = Oe^{tx} = e^{\frac{-D^2}{2}}\sum_{m=0}^{n}\frac{t^m x^m}{m!} = \sum_{m=0}^{n}\frac{t^m}{m!}\mathbb{H}_{e_m} \tag{98}$$

Recall that the $e^{tx}$ are eigenfunctions of the operator $e^{\frac{-D^2}{2}}$

$$e^{\frac{-D^2}{2}}e^{tx} = \left(1 - \frac{D^2}{2} + \ldots\right)e^{tx} = e^{tx} - \frac{t^2}{2}e^{tx} + \ldots = e^{\frac{-t^2}{2}}e^{tx}$$

Therefor we can replace $D$ by $t$ in equation (98)

$$g(x,t) = e^{\frac{-t^2}{2}}e^{tx} = \sum_{m=0}^{n}\frac{t^m}{m!}\mathbb{H}_{e_m}$$

$$g(x,t) = e^{\frac{-t^2}{2}+tx} = \sum_{m=0}^{n}\frac{t^m}{m!}\mathbb{H}_{e_m}$$

This yields the Hermite polynomial generating function.

### 3.2 Associated Lie Algebra of Laguerre Differential Operator

For Laguerre polynomials the similarity transformation of the original basis of $\mathfrak{sl}(2,R)$ will be obtained by operators in equation (42):

$$O_n = \frac{1}{n!}(D-1)^n$$

For global transformation respect to the definition, we have:

$$O = \sum_n O_n P_n = \sum_n \frac{1}{n!}(D-1)^n P_n$$

$$Y_1 = O\boldsymbol{h}O^{-1}\ ,\ Y_2 = O\boldsymbol{e}\,O^{-1}\ ,\ Y_3 = O\boldsymbol{f}\,O^{-1}$$

These generators construct a Lie algebra $\mathfrak{L}_L$ isomorphic to both $\mathfrak{sl}(2,R)$ and $\mathfrak{L}_H$. Replacing $\boldsymbol{h},\boldsymbol{e},\boldsymbol{f}$ respect to (68) we get:

$$Y_1 = O(xD-\tfrac{n}{2})\,O^{-1}\ ,\ Y_2 = ODO^{-1}\ ,\ Y_3 = O(x^2D-nx)\,O^{-1}$$

For $Y_1$ due to equation (36), substituting $\mathfrak{D}$ by $(xD-\frac{n}{2})$ and $\mathfrak{D}'$ by $\mathfrak{D}'_L-\frac{n}{2}$ simply we obtain:

$$Y_1 = O\left(xD-\frac{n}{2}\right)O^{-1} = OxDO^{-1}-\frac{n}{2} = \mathfrak{D}'_L-\frac{n}{2} \qquad (99)$$

Because of complex structures of $O$ and $P_n$, we calculate the raising operator by recursive relation:

$$(n-xD)L_n = nL_{n-1}$$

We use the operator $n-xD$ as a lowering operator $A^-$. Substitution of $L_n$ by $Ox^n$ gives rise to:

$$(n-xD)Ox^n = nL_{n-1}$$

Multiplying both side from the left by $O^{-1}$:

$$O^{-1}(n-xD)Ox^n = nO^{-1}L_{n-1}$$

$$[O^{-1}(n-xD)O]x^n = nx^{n-1}$$

Action of left side on $x^n$ equals the derivative of $x^n$, then we have:

$$O^{-1}(n-xD)O \cong D$$

Or
$$n-xD = ODO^{-1}$$

The left side should be replaced by its operator equivalent i.e.,

$$(n-xD)L_n = (\mathfrak{D}'_L-xD)L_n$$

Thus:
$$Y_2 = ODO^{-1} = \mathfrak{D}'_L-xD$$

For $Y_3$ we need $OxO^{-1}$:

$$(OxO^{-1})(ODO^{-1}) = OxDO^{-1} = \mathfrak{D}'_L$$

$$(OxO^{-1})(\mathfrak{D}'_L-xD) = \mathfrak{D}'_L$$

Or
$$OxO^{-1} = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1} \qquad (100)$$

And for $Ox^2DO^{-1}$:

$$Ox^2DO^{-1} = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}\mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}(\mathfrak{D}'_L-xD)$$

$$Ox^2DO^{-1} = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}\mathfrak{D}'_L$$

Then $Y_3$ reads as:

$$Y_3 = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}\mathfrak{D}'_L-n\mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}$$

$$Y_3 = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}(\mathfrak{D}'_L-n)$$

This operator acts as raising operator. Eventually for representation of $\mathfrak{sl}(2,R)$ in basis of Laguerre polynomial and related differential is an algebra $\mathfrak{L}_L$ with generators:

$$Y_1 = \mathfrak{D}'_L-\frac{n}{2}\quad ,\quad Y_2 = \mathfrak{D}'_L-xD\quad ,\quad Y_3 = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)^{-1}(\mathfrak{D}'_L-n) \qquad (101)$$

To prove the isomorphism of $\mathfrak{L}_L$ and $\mathfrak{sl}(2,R)$, first, we calculate the commutation relation $[Y_1\,,Y_2\,]$:

$$[Y_1\,,Y_2\,] = \left(\mathfrak{D}'_L-\frac{n}{2}\right)(\mathfrak{D}'_L-xD)-(\mathfrak{D}'_L-xD)\left(\mathfrak{D}'_L-\frac{n}{2}\right)$$

$$[Y_1\,,Y_2\,] = \mathfrak{D}'_L(\mathfrak{D}'_L-xD)-(\mathfrak{D}'_L-xD)\mathfrak{D}'_L$$

$$[Y_1\,, Y_2\,] = -\mathfrak{D}'_L xD + xD\mathfrak{D}'_L$$

We know: $$[xD^2 + D\,, xD] = xD^2 + D \tag{102}$$

Because $-\mathfrak{D}'_L = xD^2 + D - xD$, after substitution in(100) we have:

$$[-\mathfrak{D}'_L + xD\,, xD] = [-\mathfrak{D}'_L\,, xD\,] = xD^2 + D = -\mathfrak{D}'_L + xD = -Y_2 \tag{103}$$

Or: $$[Y_1\,, Y_2\,] = -Y_2$$

This is compatible with $\mathfrak{sl}(2, R)$ algebra.

For $[Y_2\,, Y_3\,]$ we have:

$$Y_2Y_3 = (\mathfrak{D}'_L - xD)[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] \tag{104}$$

$$Y_2Y_3 = \mathfrak{D}'_L[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] - xD[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)]$$

Respect to (101) and (103), in second term, substitution of $xD\mathfrak{D}'_L$ by $-Y_2+\mathfrak{D}'_L xD$ yields:

$$\begin{aligned} xD[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] &= (\mathfrak{D}'_L xD - Y_2)[(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] \\ &= [\mathfrak{D}'_L xD - (\mathfrak{D}'_L - xD)][(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] \\ &= [\mathfrak{D}'_L xD - (\mathfrak{D}'_L - xD)][(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] \\ &= \mathfrak{D}'_L xD[(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] - (\mathfrak{D}'_L - n) \end{aligned}$$

Replacing second term of (104) by this, yields:

$$\begin{aligned} Y_2Y_3 &= \mathfrak{D}'_L[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] - \mathfrak{D}'_L xD[(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] + (\mathfrak{D}'_L - n) \\ &= \mathfrak{D}'_L\{[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] - xD[(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)]\} + (\mathfrak{D}'_L - n) \\ &= \mathfrak{D}'_L\{[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)] - xD[(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)]\} + (\mathfrak{D}'_L - n) \\ &= \mathfrak{D}'_L(\mathfrak{D}'_L - xD)(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n) + (\mathfrak{D}'_L - n) = (\mathfrak{D}'_L + 1)(\mathfrak{D}'_L - n) \end{aligned}$$

$$Y_2Y_3 = (\mathfrak{D}'_L + 1)(\mathfrak{D}'_L - n) = {\mathfrak{D}'_L}^2 - (n-1)\mathfrak{D}'_L - n$$

For $Y_3Y_2$ we have:

$$\begin{aligned} Y_3Y_2 &= [\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - n)](\mathfrak{D}'_L - xD) = [\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}\mathfrak{D}'_L(\mathfrak{D}'_L - xD)] \\ &\quad - n[\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(\mathfrak{D}'_L - xD)] \end{aligned}$$

$$Y_3Y_2 = [\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}\mathfrak{D}'_L(\mathfrak{D}'_L - xD)] - n\mathfrak{D}'_L$$

Replacement of $\mathfrak{D}'_L(\mathfrak{D}'_L - xD)$ with relations of $[Y_1\,, Y_2\,]$ gives:

$$Y_3Y_2 = [\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}(-Y_2 + (\mathfrak{D}'_L - xD)\mathfrak{D}'_L)] - n\mathfrak{D}'_L$$

$$Y_3Y_2 = [\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}((\mathfrak{D}'_L - xD)(\mathfrak{D}'_L - 1)] - n\mathfrak{D}'_L$$

$$Y_3Y_2 = [\mathfrak{D}'_L(\mathfrak{D}'_L - 1)] - n\mathfrak{D}'_L = \mathfrak{D}'_L(\mathfrak{D}'_L - 1 - n) = {\mathfrak{D}'_L}^2 - (n+1)\mathfrak{D}'_L$$

Thus :

$$[Y_2\,, Y_3\,] = Y_2Y_3 - Y_3Y_2 = 2\mathfrak{D}'_L - n = 2\left(\mathfrak{D}'_L - \frac{n}{2}\right) = 2Y_1 \tag{105}$$

This proves isomorphism of $\mathfrak{L}_L$ and $\mathfrak{sl}(2, R)$ as expected.

**Lowering and Raising operators of Laguerre Polynomials and its Generating function**

Applying the method used to derive lowering and raising operators for Hermite polynomial could be repeated for Laguerre polynomials too. Respect to the properties of Lie algebra $\mathfrak{L}_L$, the generator $Y_2$ acts as lowering operator $A^-$ and $Y_3$ acts as raising operator $A^+$ on the weight vectors $\mathbb{L}_n$ which are the eigenfunctions of $Y_1$ or $\mathfrak{D}'_L$:

$$Y_2\mathbb{L}_n = (\mathfrak{D}'_L - xD)\mathbb{L}_n = n\mathbb{L}_n - xD\mathbb{L}_n$$

$$Y_2\mathbb{L}_n = (n - xD)\mathbb{L}_n = n\mathbb{L}_{n-1} \tag{106}$$

The action of $Y_1Y_2$ on $\mathbb{L}_n$ is also a lowering operator:

$$Y_1Y_2\mathbb{L}_n = (\mathfrak{D}'_L - \tfrac{n}{2})(\mathfrak{D}'_L - xD)\mathbb{L}_n$$

$$
\begin{aligned}
Y_1Y_2\mathbb{L}_n &= (\mathfrak{D}'_L - \tfrac{n}{2})(n - xD)\mathbb{L}_n \\
&= n(\mathfrak{D}'_L - \tfrac{n}{2})\mathbb{L}_{n-1} \\
&= n(\tfrac{n}{2} - 1)\mathbb{L}_{n-1}
\end{aligned} \tag{107}
$$

To derive raising operator due to the equation (100) we have:

$$\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1} = OxO^{-1}$$

Action of both side on $\mathbb{L}_n$ gives:

$$\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}\mathbb{L}_n = Oxx^n = Ox^{n+1} = \mathbb{L}_{n+1} \tag{108}$$

Thus, the operator $\mathfrak{D}'_L(\mathfrak{D}'_L - xD)^{-1}$ acts as the raising operator $A^+$ in weight vector space of Laguerre polynomials.

**Proposition 3.1**

The generating function of Laguerre polynomial is derived by projection operator method .

**proof**

due to umbral properties of operator $O = \sum_n O_n P_n$, as we proved in theorem 2.3, we have:

$$
\begin{aligned}
g(x,t) = O(1 + xt + x^2t^2 + \cdots) &= \textstyle\sum_n O_n P_n\,(1 + xt + x^2t^2 + \cdots) \\
&= \textstyle\sum_n O_n\, x^n t^n = \sum_n \mathbb{L}_n\, t^n
\end{aligned} \tag{109}
$$

Substitution the series in $xt$ powers with $\frac{1}{1-xt}$ and the identity $e^x D^n e^{-x} = (D-1)^n$ gives

$$
\begin{aligned}
g(x,t) = \textstyle\sum_n O_n P_n \frac{1}{1-xt} &= \textstyle\sum_n \frac{1}{n!} e^x D^n e^{-x} (P_n \frac{1}{1-xt}) \\
&= \textstyle\sum_n \frac{1}{n!} e^x D^n e^{-x} (P_n \frac{t^n u^{-n}}{1-xu})
\end{aligned} \tag{110}
$$

If $[u^0]$ denoted as extractor coefficient operator for $u^0 = 1$ , Then the term $P_n \frac{t^n u^{-n}}{1-xu}$ is equivalent to

$$[u^0]\frac{t^n u^{-n}}{1-xu} = P_n \frac{t^n u^{-n}}{1-xu}$$

This yields

$$
\begin{aligned}
g(x,t) &= \textstyle\sum_n \frac{1}{n!} e^x D^n e^{-x}\, [u^0]\frac{t^n u^{-n}}{1-xu} \\
&= e^x \textstyle\sum_n \frac{1}{n!} (\frac{t}{u})^n D^n\, e^{-x} [u^0]\frac{1}{1-xu}
\end{aligned}
$$

Respect to Tylor series

$$f(x+\alpha) = f(x) + \alpha f'(x) + \frac{\alpha^2}{2!} f''(x) + \cdots$$

We get

$$g(x,t) = e^x \textstyle\sum_n \frac{1}{n!} (\frac{t}{u})^n D^n\, e^{-x}[u^0]\frac{1}{1-xu} = e^x\, e^{-(x+{t}/{u})}[u^0]\frac{1}{1-(x+{t}/{u})u}$$

$$g(x,t) = [u^0]\frac{e^{(-{t}/{u})}}{1-(x+{t}/{u})u} = \frac{1}{1-t}[u^0]\frac{e^{(-{t}/{u})}}{1-\frac{xu}{1-t}}$$

Expansion of the right side in terms of $u$ with some algebra results in Laguerre generating function

$$g(x,t) = \textstyle\sum_n \mathbb{L}_n\, t^n = \frac{1}{1-t}\, e^{(\frac{-xt}{1-t})}$$

**3.3 Associated Lie Algebra of Legendre Differential Operator**

The main difference between Legendre differential operator and Hermite or Laguerre differential operator is its eigenvalues. For Hermite and Laguerre differential operators the eigenvalue are the same as the eigenvalues of original differential operator $xD$. The eigenvalues of $xD$ are integers $n$ Correspond to eigenfunctions $x^n$. The Hermite and Laguerre differential operators have the same eigenvalues and therefore we can apply the similarity transformation $OxDO^{-1}$ to derive both operators from $xD$. Note that operator $O$ is defined specific for each differential operator. For Legendre differential operator the eigenvalues are $n(n+1)$ which differs from eigenvalues of operator $\mathfrak{D}=\frac{x^2-1}{2x}D$ whose eigenvalues are integers $n$ and eigenfunctions are $(x^2-1)^n$ . In this case we alter the original operator $\mathfrak{D}$ to turn the same eigenvalues $n(n+1)$. This allows us to use similarity transformation $O\mathfrak{D}O^{-1}$ to construct Legendre associated Lie algebra isomorphic to $\mathfrak{sl}(2,R)$. Let to add $n^2$ to $\mathfrak{D}$ and act the result on the original basis$(x^2-1)^n$.

$$(\mathfrak{D}+n^2)(x^2-1)^n=\left[\frac{x^2-1}{2x}D+n^2\right](x^2-1)^n=n(n+1)(x^2-1)^n \quad (111)$$

Therefor we choose $\mathfrak{D}+n^2$ for similarity transformation of the form $O(\mathfrak{D}+n^2)O^{-1}$. Now we search for a Lie algebra $\mathfrak{L}_P$ isomorphic to $\mathfrak{sl}(2,R)$ algebra with generators to be defined based on Legendre differential operators. We define the following generators for Lie Algebra of Legendre Differential Operator.

$$Z_1=O\boldsymbol{h}'O^{-1}\ ,\ Z_2=O\boldsymbol{e}'\,O^{-1}\ ,\ Z_3=O\boldsymbol{f}'\,O^{-1}$$

The generators $\boldsymbol{h}'$, $\boldsymbol{e}'$, $\boldsymbol{f}'$ are different from $\boldsymbol{h}$, $\boldsymbol{e}$, $\boldsymbol{f}$ defined for $\mathfrak{sl}(2,R)$ in previous sections. These operators are defined to be compatible for original basis $(x^2-1)^n$. An isomorphic algebra to $\mathfrak{sl}(2,R)$ with generators $\boldsymbol{h}'$, $\boldsymbol{e}'$, $\boldsymbol{f}'$ represented as

$$\boldsymbol{h}'=\frac{x^2-1}{2x}D+n^2 \quad ,\ \boldsymbol{e}'=\frac{D}{2x} \quad ,\boldsymbol{f}'=\frac{(x^2-1)^2}{2x}D-n((x^2-1) \quad (112)$$

The commutation relations of these basis are:

$$[\boldsymbol{h}',\boldsymbol{e}']=\left(\frac{x^2-1}{2x}D+n^2\right)\frac{D}{2x}-\frac{D}{2x}\left(\frac{x^2-1}{2x}D+n^2\right)=\frac{1}{4}\left[xD\left(\frac{1}{x}\right)D-\frac{1}{x}D(xD)\right]=-\frac{D}{2x}=-\boldsymbol{e}'$$

For $[\boldsymbol{h}',\boldsymbol{f}']$ we use the identity

$$\boldsymbol{h}'=\frac{1}{x^2-1}\boldsymbol{f}'+n+n^2$$

$$[\boldsymbol{h}',\boldsymbol{f}']=\left[\frac{1}{x^2-1}\boldsymbol{f}'+n+n^2\ ,\ \boldsymbol{f}'\right]=\left[\frac{1}{x^2-1}\boldsymbol{f}'\ ,\ \boldsymbol{f}'\right]+[n+n^2\ ,\ \boldsymbol{f}']=\left[\frac{1}{x^2-1}\boldsymbol{f}'\ ,\ \boldsymbol{f}'\right]$$

$$[\boldsymbol{h}',\boldsymbol{f}']=\left(\frac{1}{x^2-1}\boldsymbol{f}'-\boldsymbol{f}'\frac{1}{x^2-1}\right)\boldsymbol{f}'$$

Some algebra shows

$$[\boldsymbol{h}',\boldsymbol{f}']=\boldsymbol{f}'$$

With these commutation relations, respect to Jacobi identity we have

$$[\boldsymbol{e}',\boldsymbol{f}']=2\boldsymbol{h}'$$

This proves that generators $\boldsymbol{h}'$, $\boldsymbol{e}'$, $\boldsymbol{f}'$ gives an isomorphic algebra to $\mathfrak{sl}(2,R)$. Based on these basis and conjugation them with operator $O$ which is defined for Legendre polynomials in equation(71), we could derive its adjoint algebra with basis that are formed by Legendre differential operator. Due to (34) and common eigenvalues of and $\boldsymbol{h}'$and $\mathfrak{D}'_{\mathcal{L}}$ (not be confused with $\mathfrak{D}'_L$ for Laguerre differential operator) we have

$$Z_1=O\boldsymbol{h}'O^{-1}=O\left(\frac{x^2-1}{2x}D+n^2\right)O^{-1}=\mathfrak{D}'_{\mathcal{L}}+n^2 \quad (113)$$

For another basis it is required to calculate $O(x^2-1)O^{-1}$. The action of this operator on Legendre polynomial $\mathbb{P}_n$ gives

$$O(x^2-1)O^{-1}\mathbb{P}_n=O(x^2-1)(x^2-1)^n=O(x^2-1)^{n+1}=\mathbb{P}_{n+1}$$

This implies that $O(x^2-1)O^{-1}$ acts as raising operator and is equivalent to $\boldsymbol{f}'$

$$\boldsymbol{f}' = O(x^2-1)O^{-1}$$

This equation and (113) gives

$$\mathfrak{D}_L' = O\left(\frac{x^2-1}{2x}D + n^2\right)O^{-1} = O\left(\frac{x^2-1}{2x}D\right)O^{-1} + n^2 = O(x^2-1)O^{-1}O\left(\frac{1}{2x}D\right)O^{-1} + n^2$$

$$\mathfrak{D}_L' = \boldsymbol{f}'(O\boldsymbol{e}'O^{-1}) + n^2$$

Or $\quad Z_2 = O\boldsymbol{e}'O^{-1} = \boldsymbol{f}'^{-1}(\mathfrak{D}_{\mathcal{L}}' - n^2) \qquad (114)$

For $Z_3$ respect to (113) we have

$$Z_3 = O\boldsymbol{f}'O^{-1} = O\left[\frac{(x^2-1)^2}{2x}D - n((x^2-1)\right]O^{-1}$$

$$Z_3 = O(x^2-1)O^{-1}O\left(\frac{x^2-1}{2x}D\right)O^{-1} - nO(x^2-1)O^{-1}$$

$$Z_3 = \boldsymbol{f}'(\mathfrak{D}_{\mathcal{L}}' - n^2) - n\boldsymbol{f}' = \boldsymbol{f}'[\mathfrak{D}_{\mathcal{L}}' - n(n+1)]$$

Thus, the set of generators for Lie algebra of Legendre differential operator are as follows

$$Z_1 = \mathfrak{D}_{\mathcal{L}}' + n^2\,, \qquad Z_2 = \boldsymbol{f}'^{-1}(\mathfrak{D}_{\mathcal{L}}' - n^2)\,, \qquad Z_3 = \boldsymbol{f}'[\mathfrak{D}_{\mathcal{L}}' - n(n+1)] \qquad (115)$$

**3.4 Adjoint representation of $\mathfrak{sl}(2,c)$ based on Hermite differential operator**

An appropriate representation of $\mathfrak{sl}(2,c)$ algebra presented as [8]:

$$\boldsymbol{h} = \frac{1}{2}xD + \frac{1}{2}\,, \quad \boldsymbol{e} = \frac{i}{2}D^2\,, \quad \boldsymbol{f} = \frac{i}{2}x^2 \qquad (116)$$

The commutation relations of these generators will be unchanged after omitting the imaginary $\boldsymbol{i}$ from $\boldsymbol{e}$ and $\boldsymbol{f}$ yields a representation of $\mathfrak{sl}(2,R)$ with commutation relations of equation (79):

$$\boldsymbol{h} = \frac{1}{2}xD + \frac{1}{2}\,, \quad \boldsymbol{e} = \frac{1}{2}D^2\,, \quad \boldsymbol{f} = \frac{1}{2}x^2$$

The adjoint representation of elements of this Lie algebra, can be derived by conjugation with any element of the group $SL(2,R)$:

$$\mathrm{Ad}_g(X) = gXg^{-1}\,, \qquad g \in SL(2,R)$$

The element $g$ could be derived by exponential map of generators of $\mathfrak{sl}(2,c)$:

$$g = e^{tX}$$

assume $X = \frac{1}{2}D^2$ and $t = -1$, then the adjoint representation elements will read as:

$$\mathrm{Ad}\,(\boldsymbol{h}) = e^{\frac{-D^2}{2}}\boldsymbol{h}e^{\frac{D^2}{2}}\,, \quad \mathrm{Ad}\,(\boldsymbol{e}) = e^{\frac{-D^2}{2}}\boldsymbol{e}\,e^{\frac{D^2}{2}}\,, \ \mathrm{Ad}\,(\boldsymbol{f}) = e^{\frac{-D^2}{2}}\boldsymbol{f}e^{\frac{D^2}{2}} \qquad (117)$$

Respect to equations (81) to (88):

$$\mathrm{Ad}(\boldsymbol{h}) = e^{\frac{-D^2}{2}}\left(\frac{1}{2}xD + \frac{1}{2}\right)e^{\frac{D^2}{2}} = \frac{1}{2}\mathfrak{D}_H' + \frac{1}{2} \qquad (118)$$

$$\mathrm{Ad}(\boldsymbol{e}) = e^{\frac{-D^2}{2}}\left(\frac{1}{2}D^2\right)e^{\frac{D^2}{2}} = \frac{1}{2}D^2$$

$$\mathrm{Ad}(\boldsymbol{f}) = e^{\frac{-D^2}{2}}\left(\frac{1}{2}x^2\right)e^{\frac{D^2}{2}} = \frac{1}{2}(x-D)^2$$

The eigenfunctions of $\boldsymbol{h}$ as Cartan subalgebra of $\mathfrak{sl}(2,R)$ are $x^n$. After conjugation with $e^{\frac{-D^2}{2}}$, the adjoint representation's Cartan subalgebra will be $\frac{1}{2}\mathfrak{D}_H' + \frac{1}{2}$ with eigenfunctions or weight vectors $\frac{1}{2}\mathbb{H}_{e_n}$. The transformation of $x^n$ to $\mathbb{H}_{e_n}$, respect to (39) is given by the relation:

$$H_{en} = e^{\frac{-D^2}{2}}x^n = gx^n \qquad g \in SL(2,R)$$

Therefore, the conjugation of generators of algebra $\mathfrak{sl}(2,R)$ by an element group $g$, results in an isomorphic adjoint algebra that its Cartan subalgebra's weight vectors (eigenfunctions) could be derived by action of the same group element on the eigenfunctions of the original Lie algebra i.e., $x^n$.

If we choose the exponent of generator $\boldsymbol{f}$ as group element $g = e^{\frac{x^2}{2}} \in SL(2,R)$ we have:

$$\mathrm{Ad}(\boldsymbol{h}) = e^{\frac{x^2}{2}}\left(\frac{1}{2}xD + \frac{1}{2}\right)e^{\frac{-x^2}{2}} = e^{\frac{x^2}{2}}(xD)e^{\frac{-x^2}{2}} + \frac{1}{2}$$

Due to Example 2.2 : $\frac{1}{2}e^{\frac{x^2}{2}}(xD)e^{\frac{-x^2}{2}} + \frac{1}{2} = \frac{1}{2}\left(e^{\frac{x^2}{2}}xe^{\frac{-x^2}{2}}\right)\left(e^{\frac{x^2}{2}}De^{\frac{-x^2}{2}}\right) + \frac{1}{2} = \frac{1}{2}x(D-x) + \frac{1}{2}$

This implies that the weight vectors of adjoint algebra should be $v_n = x^n e^{\frac{x^2}{2}}$. And can be verified by the action of $x(D-x)$ on $v_n$.

For $g = e^{\frac{tx^2}{2}}$ we get:

$$e^{\frac{tx^2}{2}}xDe^{\frac{-tx^2}{2}} = \left(e^{\frac{tx^2}{2}}xe^{\frac{-tx^2}{2}}\right)e^{\frac{tx^2}{2}}\left(-xte^{\frac{-tx^2}{2}} + e^{\frac{-tx^2}{2}}D\right) = x(-xt+D)$$

And $$e^{\frac{tx^2}{2}}xDe^{\frac{-tx^2}{2}} = x(D-xt)$$

Thus, the eigenfunctions of this operator would be $v_n = x^n e^{\frac{tx^2}{2}}$.

### 3.5 Representation of $\mathfrak{su}(2)$ and Hermite differential operator

Let introduce the basis $\mathfrak{a}_1$ , $\mathfrak{a}_2$ , $\mathfrak{a}_3$ of $\mathfrak{su}(2)$ given by

$$\mathfrak{a}_1 = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \quad \mathfrak{a}_2 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \quad \mathfrak{a}_3 = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \tag{119}$$

With commutation relations

$$[\mathfrak{a}_2, \mathfrak{a}_1] = \mathfrak{a}_3 \quad , \quad [\mathfrak{a}_3, \mathfrak{a}_2] = \mathfrak{a}_1 \quad , \quad [\mathfrak{a}_1, \mathfrak{a}_3] = \mathfrak{a}_2$$

These commutation relations coincide the complexified algebra of $\mathfrak{su}(2)$ that is the same as complexified $\mathfrak{sl}(2,R)$.

Comparing these basis with the generators of $\mathfrak{sl}(2,R)$ presented in (78) reveals the relations

$$\mathfrak{a}_1 = 2H \quad , \quad \mathfrak{a}_2 = (X+Y) \quad , \quad \mathfrak{a}_3 = (X-Y) \tag{120}$$

Conjugation of these basis with an element of the group $SL(2,R)$ gives the adjoint representation of $\mathfrak{sl}(2,R)$. Let use the operator introduced in (41) to derive Hermite polynomials from monomials $x^n$. The similarity transformations

$$X_1' = O\mathfrak{a}_1O^{-1} \qquad X_2' = O\mathfrak{a}_2O^{-1} \qquad X_3' = O\mathfrak{a}_3O^{-1} \tag{121}$$

$$X_1' = 2e^{\frac{-D^2}{2}}He^{\frac{D^2}{2}} \ , \quad X_2' = e^{\frac{-D^2}{2}}(X+Y)e^{\frac{D^2}{2}} \ , \quad X_3' = e^{\frac{-D^2}{2}}(X-Y)\,e^{\frac{D^2}{2}}$$

Substituting the basis $H, X, Y$ by (78) and (79) gives

$$X_1' = 2e^{\frac{-D^2}{2}}(xD)e^{\frac{D^2}{2}} \ , \quad X_2' = e^{\frac{-D^2}{2}}(D + x^2D - nx)e^{\frac{D^2}{2}} \ , \quad X_3' = e^{\frac{-D^2}{2}}(D - x^2D + nx)e^{\frac{D^2}{2}}$$

Thus, by equations (82) to (87) we get

$$X_1' = 2(\mathfrak{D}_H' - \frac{n}{2}) \ , \ X_2' = D + (x-D)(\mathfrak{D}_H' - n) \ , \ X_3' = D - (x-D)(\mathfrak{D}_H' - n) \tag{122}$$

The commutation relations of these basis coincide the complexified algebra of $\mathfrak{su}(2)$ and as well $\mathfrak{so}(3)$, the algebra of rotation group in 3-dimensional space.

### 3.6 General form of differential-operator representations of $\mathfrak{sl}(2, \boldsymbol{R})$

**Theorem 3.1**

Denote by $B(x)$ any function of $x$ and choose a set of its ordered integer exponents as linearly independent basis $[1\ , B(x),\ B^2(x)\ , \dots\ , B^n(x)]$, then the set of generators

$$\boldsymbol{h} = \frac{B}{B\prime}D - \frac{n}{2}\ , \qquad \boldsymbol{e} = \frac{D}{B\prime}\ , \qquad \boldsymbol{f} = \frac{B^2}{B\prime}D - nB \tag{123}$$

Satisfy the commutation relations of $\mathfrak{sl}(2, R)$ and yields an isomorphic algebra to it.

**Proof**

$$[\boldsymbol{h}\ , \boldsymbol{e}\ ] = \left(\frac{B}{B'}D\right)\frac{D}{B'} - \frac{D}{B'}\left(\frac{B}{B'}D\right)$$

$$\frac{B}{B'}\left(\frac{DB'-B''}{B'^2}D + \frac{D^2}{B\prime}\right) - \frac{1}{B'}\left(\frac{B'^2-B''\,B}{B'^2}D + \frac{B}{B'}D^2\right) = -\frac{D}{B'} = -\boldsymbol{e}$$

$$[\boldsymbol{h}\ , \boldsymbol{f}] = \frac{B}{B\prime}D\left(\frac{B^2}{B\prime}D - nB\right) - \left(\frac{B^2}{B\prime}D - nB\right)\frac{B}{B'}D = \frac{B^2}{B\prime}D - nB = \boldsymbol{f}$$

By Jacobi identity, these two commutation relations imply the third commutation relation

$$[\boldsymbol{e}\ , \boldsymbol{f}\ ] = \boldsymbol{2h}$$

Thus, the above generators are representation of the algebra $\mathfrak{sl}(2, R)$ based on an arbitrary linearly independent basis $[1\ , B(x),\ B^2(x)\ , \dots\ , B^n(x)]$ of polynomial space.

Assume these basis be transformed to new linearly independent basis $\mathcal{P}_n$ by the equation

$$\mathcal{P}_n = OB^n(x) \tag{124}$$

Where, $O$ denoted as an operator that introduced in proposition (2.1 ) and equation (8) i.e., $O = \sum_j O_j P_j$ acts on $B^n(x)$ as the n-th power of $B(x)$. Associated algebra of polynomials $\mathcal{P}_n$ can be derived as the similarity transformation or adjoint representation of $\mathfrak{sl}(2, R)$ as defined in examples. Note that the corresponding differential operator $\mathfrak{D}_{\mathcal{P}}$ is derived by $\mathfrak{D}_{\mathcal{P}} = O\boldsymbol{h}O^{-1}$. The generators of related associated algebra are

$$\mathfrak{X}_1 = O\boldsymbol{h}O^{-1}\quad , \qquad \mathfrak{X}_2 = O\boldsymbol{e}O^{-1}\quad , \qquad \mathfrak{X}_3 = O\boldsymbol{f}O^{-1} \tag{125}$$

In this setting $OBO^{-1}$ can be acts as a raising operator for $\mathcal{P}_n$basis

$$OBO^{-1}\mathcal{P}_n = OBB^n = OB^{n+1} = \mathcal{P}_{n+1}$$

Therefor we could apply this operator as raising operator $A^+$.

$$OBO^{-1} = A^+ \tag{126}$$

By this substitution, The general form of generators could be derived

$$\mathfrak{X}_1 = O\boldsymbol{h}O^{-1} = O\left(\frac{B}{B\prime}D - \frac{n}{2}\right)O^{-1} = \mathfrak{D}_{\mathcal{P}} - \frac{n}{2}$$

$$\mathfrak{D}_{\mathcal{P}} = OBO^{-1}O\frac{D}{B'}O^{-1} = A^+O\frac{D}{B'}O^{-1}$$

$$(A^+)^{-1}\mathfrak{D}_{\mathcal{P}} = O\frac{D}{B'}O^{-1} \tag{127}$$

Consequently, respect to (123) and (125) for $\mathfrak{X}_2$ we get

$$\mathfrak{X}_2 = O\boldsymbol{e}O^{-1} = O\frac{D}{B\prime}O^{-1} = (A^+)^{-1}\mathfrak{D}_{\mathcal{P}}$$

And for $\mathfrak{X}_3$

$$\mathfrak{X}_3 = OA^+O^{-1} = O\left(\frac{B^2}{B\prime}D - nB\right)O^{-1} = OBO^{-1}O\frac{B}{B\prime}DO^{-1} - nOBO^{-1}$$

$$\mathfrak{X}_3 = A^+(\mathfrak{D}_{\mathcal{P}} - n) = A^+(\mathfrak{D}_{\mathcal{P}} - n)$$

Thus, the generators

$$\mathfrak{X}_1 = \mathfrak{D}_{\mathcal{P}} - \frac{n}{2} \quad , \qquad \mathfrak{X}_2 = (A^+)^{-1}\mathfrak{D}_{\mathcal{P}} \quad , \qquad \mathfrak{X}_3 = A^+(\mathfrak{D}_{\mathcal{P}} - n) \tag{128}$$

Form an algebra $\mathfrak{L}_{\mathcal{P}}$ as a representation of $\mathfrak{sl}(2,R)$.

The polynomials $\mathcal{P}_n$ are the eigenfunctions of $\mathfrak{X}_1$ as weight vectors of Cartan subalgebra of $\mathfrak{L}_{\mathcal{P}}$.

As an example, the generators of Hermite algebra can be derived by this formula regarding the raising operator Of Hermite polynomials i.e., $A^+ = x - D$

$$X_1 = \mathfrak{D}'_H - \frac{n}{2}$$

$$X_2 = (A^+)^{-1}\mathfrak{D}'_H = (x - D)^{-1}\mathfrak{D}'_H = (x - D)^{-1}(xD - D^2) = (x - D)^{-1}(x - D)D = D$$

$$X_3 = (x - D)(\mathfrak{D}'_H - n)$$

As it is expected.

The Lie algebra $\mathfrak{L}_{\mathcal{P}}$ is the general form of representation of $\mathfrak{sl}(2,R)$ whose weight vectors are eigenfunctions of arbitrary differential operator $\mathfrak{D}_{\mathcal{P}}$. This implies that for any differential equation with eigenfunction problem, we can apply the corresponding algebra $\mathfrak{L}_{\mathcal{P}}$ and its raising operator to derive its solutions as described below.

### 3.7 Solutions to Differential equations by Raising operator method

In this section we apply the raising operators of the Lie algebra associated with differential operators defining the related differential equations to derive its solutions. We start with a known differential equation and first two solutions i.e., the first two eigenfunctions with the lowest eigenvalues. Then by the definition of raising operator $A^+$ defined by (126), we derive this operator by restriction to 2 dimension of polynomial space and using the first two terms of $\sum_j O_j P_j$ and Forbenius covariant operator, the entire eigenfunction (solutions) of the differential equation could be derived.

**Example 3.1**

As an example, for Laguerre differential equation, if we know the first two monomial i.e., $\mathbb{L}_0 = \mathbf{1}$ and $\mathbb{L}_1 = -x + 1$ as the trivial eigenfunctions, respect to equation (126) the raising operator is

$$A^+ = OBO^{-1}$$

Where operator $O$ transforms the basis $[x^n]$ to Laguerre polynomials $\mathbb{L}_n$.

For Laguerre differential equation by $B = x$, the raising operator appears as

$$A^+ = OxO^{-1} = (\textstyle\sum_j O_j P_j)xO^{-1} \tag{129}$$

$$A^+ = (O_1 P_1)xO^{-1}$$

And acting both side on $\mathbf{1}$ as the first monomial we get

$$A^+.\mathbf{1} = (O_1 P_1)xO^{-1}.\mathbf{1} \tag{130}$$

By $O^{-1}.\mathbf{1} = \mathbf{1}$ and $P_1 x = x$ and by the $O_1 = D - 1$ , this equation yields

$$A^+.\mathbf{1} = (D-1)x$$
$$\mathbf{1} = (A^+)^{-1}(D-1)x$$
The action of operator $(A^+)^{-1}(D-1)$ on $x$ is the same as $D$, then we have the identity

$$(A^+)^{-1}(D-1) = D \tag{131}$$
$$(A^+)^{-1}(D-1)D^{-1} = \mathbf{1}$$
$$(A^+)^{-1}(1-D^{-1}) = \mathbf{1}$$
And this gives
$$A^+ = 1 - D^{-1} \tag{132}$$
Applying this operator on the first two Laguerre polynomials gives the nth solution

$$\mathbb{L}_n = (A^+)^n.\mathbf{1} = (1-D^{-1})^n.\mathbf{1} \tag{133}$$
This method can be applied for any differential operator to find its eigenfunctions or ordered solutions.

**Example 3.2**

For Hermite differential equation to derive $O_1$ due to equation (37) for 2 dimension we have

$$\mathfrak{D}'_H = (\lambda'_0 O_0 P_0 + \lambda'_1 O_1 P_1)(\textstyle\sum_j OP_j)^{-1}$$
We assume $\lambda'_0 = 0$, $\lambda'_1 = 1$ and $O = (\sum_j OP_j)^{-1}$

$$\mathfrak{D}'_H = O_1 P_1 O^{-1} \tag{134}$$
Acting both side on first basis $x$ definition for projection operator $P_1$ , gives

$$\mathfrak{D}'_H\, x = O_1 P_1 O^{-1} x$$
$$\mathfrak{D}'_H\, x = O_1 P_1\, x$$
$$\mathfrak{D}'_H\, x = O_1 (xD) x$$
This equation shows both operators in the equation are equivalent

$$\mathfrak{D}'_H = O_1(xD) \tag{135}$$
Substitution for $\mathfrak{D}'_H$ and action of $D^{-1}$ on both sides, yields
$$(xD - D^2)D^{-1} = O_1(xD)D^{-1}$$
$$x - D = O_1 x$$
Or
$$(x-D)x^{-1} = O_1 \tag{136}$$
Respect to $A^+ = O_1 B O^{-1}$ we get
$$A^+ = (x-D)x^{-1}xO^{-1}$$
Acting both side on **1** as the first **1** basis
$$A^+.\mathbf{1} = (x-D)x^{-1}xO^{-1}.\mathbf{1} \tag{137}$$
$$A^+.\mathbf{1} = (x-D)x^{-1}x.\mathbf{1}$$
Thus, we have
$$A^+ = x - D$$
With raising operator, we derive all Hermits eigenfunctions as solutions to its differential equation

$$\mathbb{H}_{e_n} = (A^+)^n.\mathbf{1} = (x-D)^n.\mathbf{1} \tag{138}$$

### 3.8 Baker-Campbell-Hausdorff formula application for Lie Algebras of Differential Operators

A specific version of Baker-Campbell-Hausdorff formula implies that if the commutator relation of a Lie algebra generators $X_1, X_2$ meets the equation [6]:

$$[X_1, X_2] = s\, X_2 \tag{139}$$

With $s \in R$, then the BCH formula reduces to

$$e^{X_1} e^{X_2} = \exp\left(X_1 + \frac{sX_2}{1-e^{-s}}\right) \tag{140}$$

Adjoint representation of $\mathfrak{sl}(2, c)$ as defined in equations (116) and (118) represented by generators

$$\mathrm{Ad}(\boldsymbol{h}) = \frac{1}{2}\mathfrak{D}'_H + \frac{1}{2} \quad , \quad \mathrm{Ad}(\boldsymbol{e}) = \frac{1}{2}D^2 \quad , \quad \mathrm{Ad}(\boldsymbol{f}) = \frac{1}{2}(x-D)^2$$

That obey the commutation relations in equations (79)

$$\left[\frac{1}{2}\mathfrak{D}'_H, \frac{1}{2}D^2\ \right] = -\frac{1}{2}D^2$$

Multiplying by $-1$ yields

$$\left[\frac{1}{2}\mathfrak{D}'_H, -\frac{1}{2}D^2\ \right] = \frac{1}{2}D^2$$

Due to (139) and (140) we have

$$e^{\mathfrak{D}'_H}\, e^{-\frac{1}{2}D^2} = \exp\left(\mathfrak{D}'_H + \frac{D^2}{2(1-e)}\right)$$

Acting both sides on $x^n$ by equation (39) yields

$$e^{\mathfrak{D}'_H}\, e^{-\frac{1}{2}D^2} x^n = \exp\left(\mathfrak{D}'_H + \frac{D^2}{2(1-e)}\right) x^n$$

$$e^{\mathfrak{D}'_H}\mathbb{H}_{e_n} = \exp\left(\mathfrak{D}'_H + \frac{D^2}{2(1-e)}\right) x^n$$

$$e^{n}\mathbb{H}_{e_n} = \exp\left(\mathfrak{D}'_H + \frac{D^2}{2(1-e)}\right) x^n$$

$$\mathbb{H}_{e_n} = \exp\left(\mathfrak{D}'_H - n + \frac{D^2}{2(1-e)}\right) x^n$$

This is a new relation the converts $x^n$ to $\mathbb{H}_{e_n}$ and alternative to the classic relation:

$$\mathbb{H}_{e_n} = e^{-\frac{1}{2}D^2} x^n$$

This technique is also applicable to other differential operators such as Laguerre and Legendre differential operators.

References


1) Ismail, Mourad, Mourad EH Ismail, and Walter van Assche. *Classical and quantum orthogonal polynomials in one variable*. Vol. 13. Cambridge university press, 2005.
2) Arfken, George B., and Hans-Jurgen Weber. "Mathematical methods for physicists." (1972).
3) Rasala, Richard. "The Rodrigues formula and polynomial differential operators." *Journal of Mathematical Analysis and Applications* 84.2 (1981): 443-482.
4) Post, Gerhard, and Nico van den Hijligenberg. "gl (λ) and differential operators preserving polynomials." *Acta Applicandae Mathematica* 44 (1996): 257-268.
5) Turbiner, A. "Lie algebras and polynomials in one variable." *Journal of Physics A: Mathematical and General* 25.18 (1992): L1087.

6) Matone, Marco. "Closed form of the Baker–Campbell–Hausdorff formula for the generators of semisimple complex Lie algebras." *The European Physical Journal C* 76 (2016): 1-7.
7) Horn, Roger A., and Charles R. Johnson. "Topics in matrix analysis Cambridge university press." *Cambridge, UK* (1991).
8) Howe, Roger E., and Eng Chye Tan. *Non-Abelian Harmonic Analysis: Applications of SL (2,c)*. Springer Science & Business Media, 2012.